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L. *The Solution of Algebraic Equations by means of Tables of Logarithms.* *By* V. A. Bailey, *M.A.*, *D.Phil.* (*Oxon.*), *F.Inst.P.*, *Associate Professor of Physics, University of Sydney* *.

1. THE methods most commonly discussed in text-books † for evaluating the real roots of a real algebraic equation are those associated with the names of Newton and Horner respectively.

They involve mainly computations which ultimately are equivalent to the evaluation of terms like $p\,x^m$ (m an integer), where p and x may both contain several digits.

In general the latter evaluation is most conveniently carried out by means of logarithmic tables ‡, so it is well to examine whether a practical method can be devised for approximating to the real roots with the help of such tables; the results of such an examination are given in the following sections.

2. Let the equation whose real roots are required be

$$f(x) \equiv p_0 + p_1 x + \ldots + p_m x^m = 0, \quad . \quad . \quad (1)$$

* Communicated by the Author.

† *Cf.* works on the theory of equations by Burnside and Panton, Cajori, Dickson, and Runge and König.

‡ Which are generally more easily available than calculating machines.

where the coefficients a_r are all real, and consider first x positive.

Set $$x=10^{\xi}, \quad |p_r|=10^{\alpha_r},$$

i. e., $$\xi=\log_{10}x, \quad \alpha_r=\log|p_r|.$$

The equation then becomes

$$\phi(\xi)\equiv\sum_0^m \pm A(\alpha_r+r\xi)=0, \quad . \quad . \quad . \quad . \quad (2)$$

where $A(z)$ stands for antilog (z), and the sign before each symbol A is given by equation (1).

Therefore $$\phi'(\xi)=c\sum_1^m \pm rA(\alpha_r+r\xi),$$

where $$c=2{\cdot}302\ldots.$$

Let x_0 be an initial value of the root determined by means of any simple and convenient method. Then $\delta_0=\log_{10}x_0$ is a first approximation to the value of ξ in (2).

The successive stages of approximation $\xi_1, \xi_2, \xi_3, \ldots$ are then given by the formula

$$\xi_{n+1}=\xi_n+\delta_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where, by Newton's theorem,

$$\delta_n=-\phi(\xi_n)/\phi'(\xi_n).$$

Thus $$\xi_n=\delta_0+\delta_1+\ldots+\delta_{n-1}. \quad . \quad . \quad . \quad . \quad (4)$$

Let $$\alpha_r^{(n)}=\alpha_r+r\xi_n,$$

where (n) stands for n primes.

Therefore, by (3),

$$\alpha_r^{(n+1)}=\alpha_r^{(n)}+r\delta_n. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Also $$\phi(\xi_n)=\sum_{r=0}^m \pm A(\alpha_r+r\xi_n)$$
$$=\sum_{r=0}^m \pm A(\alpha_r^{(n)}),$$

and $$\phi'(\xi_n)=c\sum_{r=1}^m \pm rA(\alpha_r^{(n)}).$$

Thus

$$\delta_n=-\frac{\sum_{r=0}^m \pm A(\alpha_r^{(n)})}{\sum_{r=1}^m \pm rA(\alpha_r^{(n)})}\times\mu, \quad . \quad . \quad . \quad . \quad (6)$$

where $$\mu={\cdot}43429\ldots.$$

In an evident notation this may also be written as

$$c\delta_n = -\frac{\Sigma A_r^{(n)}}{\Sigma r A_r^{(n)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By taking three terms in Taylor's expansion of $\phi(\xi_n+\delta_n)$ it is easy to show that a closer approximation is obtained at each stage by using the formula

$$c\delta_n = -\frac{\Sigma A_r^{(n)}}{\Sigma r A_r^{(n)} + \frac{1}{2}(c\delta_n)\Sigma r^2 A_r^{(n)}}, \quad . \quad . \quad . \quad (8)$$

or
$$c\delta_n = \frac{-\Sigma A_r^{(n)}}{\Sigma r A_r^{(n)} + (-\Sigma A_r^{(n)})(\Sigma r^2 A_r^{(n)})/2\Sigma r A_r^{(n)}}. \quad . \quad (9)$$

$$= -\frac{\Sigma A_r^{(n)}}{D} \quad \text{say.}$$

The convergency of the process may be studied as follows :—

The correction δ_n in (8) really represents two successive correction δ_n and δ_{n+1} as defined by (7). Thus, on the left-hand side of (8) we may set $c(\delta_n+\delta_{n+1})$ in place of $c\delta_n$, but need not on the right-hand side if $\delta_{n+1}<\delta_n$.

Subtracting (7) from (8) then gives

$$c\delta_{n+1} \doteqdot \frac{(\Sigma A_r^{(n)})(\Sigma r^2 A_r^{(n)})}{(\Sigma r A_r^{(n)})^2}\,\tfrac{1}{2}(c\delta_n),$$

and so, by (7)
$$\delta_{n+1} \doteqdot \lambda\delta_n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where
$$\lambda = -\frac{c\Sigma r^2 A_r^{(n)}}{2\Sigma r A_r^{(n)}}.$$

In general λ changes little from one stage to another, so (10) leads to the relation

$$\delta_{n+1} \doteqdot \delta_n(\delta_n/\delta_{n-1})^2.$$

This shows that if δ_{n+1}/δ_n is of the order 10^{-p}, then $\delta_{n+2}/\delta_{n+1}$ is of the order 10^{-2p}, etc., and so δ_{n+t}/δ_n is of the order $10^{-p(1+2+2^2+\cdots+2^{t-1})}$, *i.e.*, of the order $10^{-p(2^t-1)}$.

This also represents the order of Δ_{n+t}/Δ_n, where $\Delta_n = x_{n+1}-x_n$ and x_n is the nth approximation to the root of equation (1).

Thus, for example, if $p=1$, then three more stages add seven more digits to the value of the root, and if $p=2$ then two more stages add six digits.

In Horner's process δ_{n+1}/δ_n is of the order 10^{-1}, *i. e.*, δ_{n+t}/δ_n is of order 10^{-t}. The last stage of Horner, involving long division alone, converts this to $10^{-2(t-1)}$ for t stages in all. Thus three stages add four more digits

3. The computations involved in the use of the formula (6) may be conveniently carried out as shown in the following scheme (Table I.).

TABLE I.

	x_0.	δ_0.	δ_1.		δ_2.		...	δ_n.		...
1 ..	$\lvert p_1 \rvert$	α_1	α'_1		α''_1		...	$\alpha_1^{(n)}$		...
2 ..	$\lvert p_2 \rvert$	α_2	α'_2		α''_2		...	$\alpha_2^{(n)}$		...
.	.	.	.		.		.	.		.
.	.	.	.		.		.	.		.
r ..	$\lvert p_r \rvert$	α_r	α'_r		α''_r		...	$\alpha_r^{(n)}$		...
.	.	.	.		.		.	.		.
.	.	.	.		.		.	.		.
0 ..	$\pm$		$\lvert p_0 \rvert$		$\lvert p_0 \rvert$		...	$\lvert p_0 \rvert$		...
1 ..	$\pm$		A'_1	A'_1	A''_1	A''_1	...	$A_1^{(n)}$	$A_1^{(n)}$	...
2 ..	$\pm$		A'_2	$2A'_2$	A''_2	$2A''_2$	...	$A_2^{(n)}$	$2A_2^{(n)}$	...
.	.		.	.	.	.	.	.	.	.
.	.		.	.	.	.	.	.	.	.
r ..	$\pm$	Σ+ve δ's	A'_r	rA'_r	A''_r	rA''_r	...	$A_r^{(n)}$	$rA_r^{(n)}$	...
.	.	Σ−ve δ's	.	.	.	.	.	.	.	.
.	.		.	.	.	.	.	.	.	.
	x	ξ	$\Sigma A'_r$	$\Sigma rA'_r$	$\Sigma A''_r$	$\Sigma rA''_r$	...	$\Sigma A_r^{(n)}$	$\Sigma rA_r^{(n)}$	...

The first column contains the order numbers 1, 2, . . . and the second column contains the coefficients $|p_1|$, $|p_2|$, . . . in the upper half and all the signs in the lower half. The logarithms α_1, α_2 . . . of the coefficients are then written in the succeeding half-column.

The initial value of the root x_0 is now set at the top of the first column and its logarithm δ_0 at the top of the second column.

The quantities α'_1, α'_2, . . . , in virtue of (5), are calculated in succession by adding δ_0 to α_1, $2\delta_0$ to α_2, $3\delta_0$ to α_3, etc., and written in the next (fourth) column. Their

antilogs A'_1, A'_2 . . . , preceded by $A'_0 = |p_0|$, are then written in the lower half of the same column. These are summed together, the sign preceding each term being given by the lower half of the second column. The sum $\Sigma A'_r$ is written in the last row. In the lower half of the fourth column the quantities A'_1, $2A'_2$, $3A'_3$ . . . are calculated to one more than the number of significant figures $\Sigma A'_r$ is found to have, and their sum $\Sigma r A'_r$ is also written in the last row.

The correction δ_1 is then determined as

$$-(\Sigma A'_r/\Sigma r A_r) \times {\cdot}4343,$$

by means of a slide-rule, and written at the head of the fourth column.

The process from the α_r's to δ_1 constitutes the first stage. In the next stage we similarly proceed from the α''_r's to δ_2, and so on.

When $\delta_n(\delta_n/\delta_{n-1})^2$ is sufficiently small we now sum δ_0, δ_1, δ_2, . . . , δ_n and so obtain ξ. The lower half of the third column may be used for this purpose. Then x, the antilog of ξ, is the required approximation to the root, and is written at the bottom of the second column.

To determine a negative root we replace x by $-x$, and then proceed as for a positive root.

4. An example of the application of this method is shown in Table II., from which it will be seen that $x = 2{\cdot}114442$ is the value, correct to 7 figures, of a root of the equation

$${\cdot}9826x^3 - 2{\cdot}601x^2 + 1{\cdot}615x - 1{\cdot}075 = 0.$$

Until $10^4 \, |\delta_n| \, (\delta_n/\delta_{n-1})^2 < 1$ we use 4-figure tables *.

The letters a, b, c, d . . . indicate the order in which the various numbers are computed. Thus

$$j = f + i, \qquad k = g + 2i, \quad l = h + 3i,$$

$$m = 1{\cdot}075,\ n = A(j), \qquad o = A(k), \quad p = A(l),$$

$$q = -m + n - \sigma + p, \qquad r = n, \qquad s = 2\sigma,\ t = 3p,$$

$$u = r - s + t, \qquad v = q, \qquad w = -(v/u) \times {\cdot}4343.$$

* Such as those in Kaye and Laby's book, or in the Mathematical Tables of Milne-Thomson and Comrie (Macmillan & Co.).

Table II.

$$\cdot 9826x^3 - 2\cdot 601x^2 + 1\cdot 615x - 1\cdot 075 = 0.$$

A root lies near root $2\cdot a$.

	2.	*b.*	·3.	*i.*	·029.	*w.*	−·0037.		Error ≐ ·0000001 −·0001041.	
1	1·615	*c*	·2081	*f*	·5081	*j*	·5371		·533 4725	
2	2·601	*d*	·4152	*g*	1·0152	*k*	1·0732		1·065 7404	
3	·9826	*e*	$\bar{1}$·9924	*h*	·8924	*l*	·9794		·968 2768	
0	−		..		1·07 *m*		1·075		1·075	
1	+		..		3·22 *n*	3·22 *r*	3·444	3·44	3·415643	3·415
2	−		·329		10·35 *o*	20·7 *s*	11·834*	23·67	11.634303	23·268
3	+		−·0038041		7·80 *p*	23·4 *t*	9·537	28·61	9·295585	27·885
	2·1144424		·3251959		$\bar{1}$·60 *q*	5·9 *u*	0·072	8·38	0·001925	8·032
					−0·40 *v*					

* Using 5-fig. logs.

The determination of the sum q is carried out as follows, counting from the right :—

In the first column :

$-7+2=-5$, and $-5=-10$; set down 0 and carry -1.

In the second column :

$-1-0=-1$, and $+2=1$, and $-3=-2$, and $+8=6$; set down 6 and carry 0.

In the third column:

$0-1=-1$, and $+3=2$, and $-10=-8$, and $+7=-1$; set down $\bar{1}$.

The number of significant figures retained in each of r, s, and t is one more than the number in q.

The number of significant figures retained in w is the same as the number in q.

At the end of the second stage we have

$$10^4 \mid \delta_2 \mid (\delta_2/\delta_1)^2 \doteqdot 0{\cdot}6,$$

so we use 7-figure tables * in the succeeding stages until

$$10^7 \mid \delta_n \mid (\delta_n/\delta_{n-1})^2 < 1.$$

At the end of the third stage

$$10^7 \mid \delta_3 \mid (\delta_3/\delta_2)^2 \doteqdot 0{\cdot}9,$$

so the error in ξ is then about 1 in the seventh decimal place.

Closer approximation would require the use of tables of more than 7-figure logarithms, but these are almost never needed in practical problems.

For an equation involving large exponents like the following,

$$3{\cdot}9\,x^{15}-8{\cdot}135\,x^{14}+1104\,x^2-205400=0,$$

we first divide it through by x^7 (say), so giving

$$3{\cdot}9\,x^8-8{\cdot}135\,x^7+1104\,x^{-5}-205400\,x^{-7}=0,$$

and then proceed as in Table III.

5. If the formula (9) is used at any stage in place of (7) a higher convergence occurs.

The scheme shown in Table I. can be easily modified at

* Such as Chambers's Tables.

TABLE III.

$$3{\cdot}9\,x^{8}-8{\cdot}135\,x^{7}+1104\,x^{-5}-205400\,x^{-7}=0.$$

$$\left.\begin{array}{l} f(2)\doteqdot-1600 \\ f(3)\doteqdot\;\;3080 \end{array}\right\}\quad\therefore\quad x_0\doteqdot 2{\cdot}3.$$

	2·3.	0·36.	·016.		−·00089 "Error" ≑ ·000003.	
−7	205400	5·3126	2·7926		2·6806	
−5	1104	3·0429	1·2429		1·1629	
7	8·135	·9104	3·4304		3·5424	
8	3·9	·5911	3·4711		3·5991	
−7	−	..	620·0	+ 4340	479	+ 3300
−5	+	..	17·5	− 87	14	
7	−	·376	2695	−18865	3486	−24400
8	+	−·00089	2959	+23672	3973	+31800
	2·3720	·37511	$\bar{1}661$ −339	9060	22	10700

any stage to embody this process. Thus the first stage would become the following :—

$\delta_1=-{\cdot}4343\Sigma A'_r/D$		
α'_1		
α'_2		
.		
.		
$\lvert p_0\rvert$		
A'_1	A'_1	A'_1
A'_2	$2A_2$	$2(2A_2)$
.	.	.
.	.	.
A'_r	rA'_r	$r(rA'_r)$
.	.	.
.	.	.
$\Sigma A'_r$	$\Sigma rA'_r$	$\Sigma r^2A'_r$
	$-\dfrac{(\Sigma A')(\Sigma r^2A')}{2\Sigma rA'}$	
D		

Each such stage is equivalent to two successive stages in the scheme of Table I., and takes little more than half as long to carry out.

It is most usefully adopted in the earlier stages of a computation, but may also be used in the last stage as a means of estimating the final error.

The result of applying this method to our first equation is shown in Table IV., and may be compared with Table II.

6. In order to make a comparison of our method with Horner's the latter has been applied to the two equations already studied.

TABLE IV.

	2.	·3.	·0250 *p*.	·0001960.
1 ..	1.615	·2081	.5081	·533 1725
2 ..	2·601	·4152	1·0152	1·065 1404
3 ..	·9826	$\bar{1}$·9924	·8924	·967 3768
0 ..	–	..	1·075 *a*	1·075
1 ..	+	..	3·222 *b* 3·22 *f* 3·2 *j*	3·413285 3·413 3·4
2 ..	–	..	10·356 *c* 20·71 *g* 41·4 *k*	11·618241 23·236 46·5
3 ..	+	·3251960	7·805 *d* 23·41 *h* 70·2 *l*	9·276342 27·829 83·5
	2·1144427	·3251960	–0·404 *e* 5·92 *i* 32.0 *m*	–·003614 8·006 40·4
			1·09 *n*	·009
			7·01 *o*	8·015

For the cubic equation the two methods involve about the same amount of work, but for the given equation of the fifteenth degree Horner's method requires many times as much work as ours.

Also a single error with Horner's method will generally lead to a wrong value for the root, but with our logarithmic method an error in the lower half of each stage is either very quickly detected by giving a value of δ_n greater than δ_{n-1} or is rectified in the succeeding stages, and merely * prolongs the computation unnecessarily. In any event, a check can be made as follows when desired:

In the third column we extend the quantities $\alpha_1, \alpha_2, \ldots$ to 7 decimal places, and write ξ_{n-1} just above δ_0 in the third column, where ξ_{n-1} is the penultimate approximation to ξ, *e. g.*, ·3253 in Table II., or ·325 in Table IV.

* Occasionally the error may also transfer the computation to that for another root.

Table V.

$$x^3-9x-2=0.$$

Roots lie near 3, $-0{\cdot}2$, and -3.

	3.	·48.	·0124.		
1 ……	9	·9542	1·4342		
2 ……	0				
3 ……	1	·0	1·44		
0 ……	—	. .	2		
1 ……	—	. .	27·20	—27·2	— 27
2 ……	0	. .			
3 ……	+	. .	27·53	82·6	248
	3·108	·4924	$\bar{1}$8·33	55·4	221
			— 1·67	+ 3·3	
				58·7	
	·2.	$\bar{1}$.3.	.0493.		
1 ……	9	·9542	·2542		
2 ……	0				
3 ……	1	·0	$\bar{3}$·9		
0 ……	—	. .	2		
1 ……	+	. .	1·795	1·795	1·795
2 ……	0	. .			
3 ……	—	. .	·00795	— ·024	—0·072
	— ·2235	$\bar{1}$·3493	$\bar{1}$·787	1·771	1·723
			— ·213	·103	
				1·874	
	3.	·48.	— ·0199.		
1 ……	9	·9542	1·434		
2 ……	0				
3 ……	1	·0	1·44		
0 ……	—	. .	2		
1 ……	+	. .	27·20	27·2	27·2
2 ……	0	. .			
3 ……	—	. .	27·53	— 82·6	—247·8
	—2·885	.4601	— 2·33	—55·4	—221
				4·6	
				—50·8	

If the usual process be now applied to this column, using ξ_{n-1} in place of δ_0, it should lead directly to the final column.

7. The method of T. Weddle (v. Workman, 'Memoranda Mathematica') is related to ours in that the root is determined as $x_0 f_1 f_2 \ldots f_n$, where the f's are a series of factors approximately equal to and converging upon 1. It involves, however, much multiplication, particularly when each coefficient contains several digits.

If a root is required correct to about three significant figures the above methods can all be applied with the help of a 10-inch slide-rule alone, since this usually carries a scale of logarithms on the back of the slide.

The determination in this way of the three roots of the equation

$$x^3-9x-2=0$$

is shown in Table V.

In conclusion, I wish to acknowledge my indebtedness to Mr. R. H. Healey for several computations required in connexion with this work, and also for the benefit of discussions with him on various points.

LI. *Magnetic Effects in Iron Crystals.* *By* A. G. Hill, *B.Sc., Ph.D., formerly Research Student, Physics Department, University College, Dundee* *.

IT was stated in a previous paper (Phil. Mag. ser. 7, xiv. p. 599, Oct. 1932) that the form of the curve of magnetization for iron crystals still remains somewhat unsettled, some workers holding that, though continuous, it may consist of knees linked by straight lines. It appeared from a study of the existing literature that end-effects necessitate corrections of such relative magnitude that the form of the final curve must always be in doubt where these are necessary, and so it seemed advisable to devise some method independent of corrections. The previous work (*loc. cit.*) at low fields over a temperature-range of $-190°$ C., to 162° C. and the present work near saturation over a temperature-range from

* Communicated by Prof. W. Peddie, D.Sc., F.R.S.E.

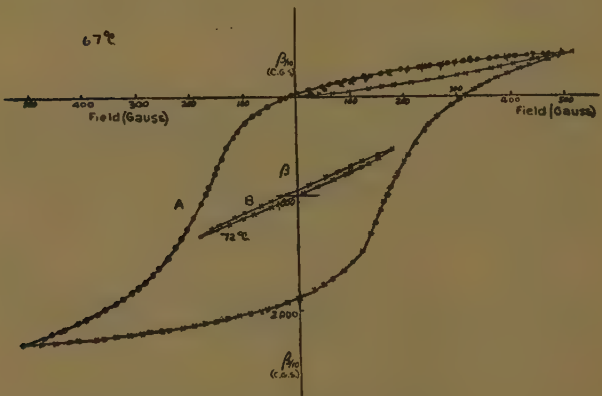
−190° C. to 525° C., was undertaken with this aim. It is necessary to employ fields approaching saturation value because the supposed kinks occur about the knee of the magnetization curve.

Observational Results.

(a) *Experiments near Saturation at Temperatures between* −190° C. *and* 312° C.

The cylindrical crystalline shell was magnetized circularly by a current passing along its axis, so that there were no ends on the specimen in the direction of magnetization.

Fig, 1.



Investigations carried out at temperatures of −190, −70 0, 67, 88, 101, 201, and 312° C., using approximately equal maximum fields of 520 gauss, gave identical results. Curve A (fig. 1) illustrates the results obtained at 67° C. There were no kinks in any part of the curves. The non-symmetry about the field-axis is due to the initial magnetized state of the crystal, and does not influence the smoothness of the curve. The initial magnetization curves from zero magnetization can, of course, be readily obtained from the experimental curves, but this treatment is not necessary.

From these investigations it was concluded that (1) there are no discontinuities in the magnetization curves obtained

by summing the magnetizations in all directions in a face-plane of an iron crystal; (2) there is a definite decrease with increasing temperature in the average values of (*a*) the maximum induction, (*b*) the coercive force, and (*c*) the residual magnetism; (3) there is a distinct decrease in the hysteresis loss with increase of temperature under the conditions specified.

(b) *Magnetization along a Cubic Axis.*

The crystal was mounted concentric with a solenoid, so that it could be magnetized along its axis. The demagnetizing field brought into action in this arrangement would not be greater than that found by previous investigations, and it was partly to investigate the influence of this field that the following experiments were carried out. It is obvious, also, that kinks, if at all in existence, should be shown more clearly in the condition of uni-directional magnetization than in the curves obtained in the preceding section. It will be noted (fig. 1, curve A) that fields of 200 gauss were sufficient to magnetize the specimen beyond the knee. The solenoid was therefore designed to produce fields of this magnitude. The experiments described in (*a*) showed that it was advisable to begin an experiment with the crystal in as standard a state as possible. This was approximated to in the following experiments by magnetizing the crystal once through a complete cycle at the given temperature before taking any readings.

Experiments were carried out at temperatures of -190, -78, 0, 72, 117, 231, 306, 383, and 525° C. The graphs obtained were only faintly curved, and the hysteresis loss was slight; there was no approach to saturation.

The ordinates in curve B (fig. 1) have been multiplied by a factor 10, and show the results obtained at 72° C. That induction in this series is only a small fraction of that obtained in (*a*) is clearly seen from the following numerical values:—

	Temp. °C.	β in C.G.S. at 100 gauss.	Ratio.	Hysteresis loss in ergs.	Ratio.
(*a*)....	72	245	1/33	18	1/22
(*b*)....	67	8000	—	402	—

These data show very definitely how great the demagnetization factor may be. A further series of experiments over the same temperature range, using fields of approximately 500 gauss, gave similar results.

It appeared that (1) self-demagnetization reduces induction and hysteresis by a large factor, and (2) that the slopes of these curves vary linearly with temperature according to the equation $4\pi \frac{dI}{dH} = 4{\cdot}29 + {\cdot}131\ T$.

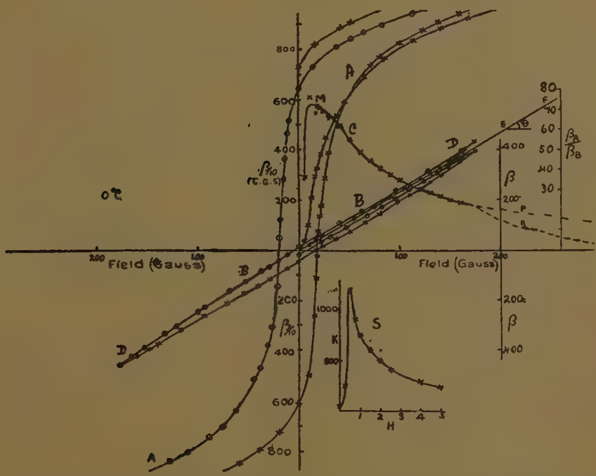
(c) *Experiments on the Magnetization of a Soft Iron Cylindrical Shell of the same Dimensions as those of the Crystal.*

In the above work we have a crystalline body magnetized (*a*) when self-demagnetization was not acting, and (*b*) when self-demagnetization was acting. A great difference in the effect is clearly demonstrated above. There still remained the possibility that the crystal was not in the same magnetic state in the two series of experiments prior to magnetization. If, for example, the crystal was in an almost saturated state at the commencement of the (*b*) series of experiments, the magnetization curves would necessarily be of the type found in these experiments. Before it is possible to attribute the decrease in induction purely to self-demagnetization, therefore, it is necessary to investigate the material in a definite initial magnetic state. A cylindrical shell of soft iron, having the same dimensions as those of the crystal, was cut from a rod and heated to red-heat. The shell was then magnetized as in (*a*), with no demagnetizing field acting upon it. It was then removed from the apparatus, heated to red-heat, reinserted, and magnetized as in (*b*), *i. e.*, with a demagnetizing field acting. The curves illustrating the results obtained are shown in fig. 2, curves A and B respectively. The ordinates of B have also been multiplied by factor 10.

The demagnetizing action is again clearly seen. The maximum induction and hysteresis loss are smaller than those obtained from the crystal, the decreases being due to the differences in the material. Curve B is very similar to those obtained when using the crystalline material. Its average slope is greater; tan θ being ·59, whereas the other slope is ·55. This increase of slope

is again due to a difference in the material. Only at the tips D is there a tendency for the induction to increase more rapidly. The ordinates of the curve C represent the ratio $\frac{\beta_A}{\beta_B}$. A sharp maximum (M) occurs at low field intensity. The value of the ratio at this point is 74. At fields of 100 gauss the ratio is 35, a value very close to the value of 33 derived in the above table. At higher fields the curve B would doubtless lose its linearity rapidly. If this occurred, curve C will follow a course like that

Fig. 2.



of the dotted line R, but if B continues to be linear then it will follow a course such as P. In either case the ultimate value of the ratio will be unity. Curve C may well be called the correction curve. It is very similar in form to the curve S showing the growth in susceptibility in initially non-magnetized iron with increasing field-strength. Similar correction curves will apply to all the experimental curves obtained in (*b*), *e.g.*, curve B in fig. 1.

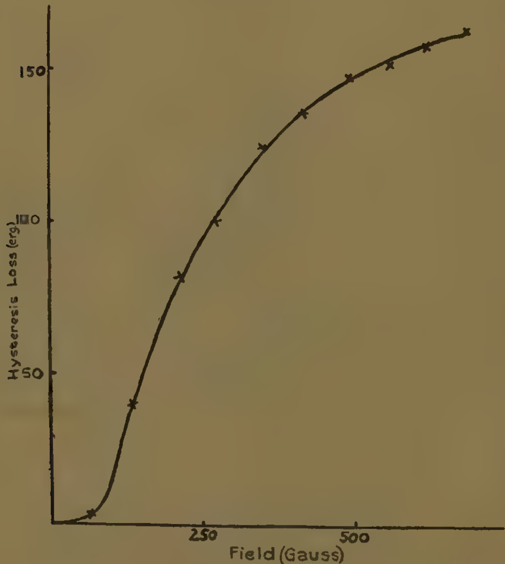
It was concluded (1) that the magnitude of the demagnetization factor of roughly homogeneous crystalline material is very great, and of the same order as that for

heated soft iron, and (2) that the low induction and decreased loss of hysteresis shown in these experiments on magnetization along a cubic axis (a) is due entirely to self-demagnetization effects.

Discussion.

We have already noted (former paper) that Foster, Bozorth, Gerlach, Sizoo, Webster, and others used

Fig. 3.



specimens necessitating corrections for self-demagnetization effects ; and that several workers were of the opinion that it is self-demagnetization that causes the kinks to appear in the magnetization curves of iron crystals.

The present results show that self-demagnetization may indeed be very strong, probably sufficiently so to produce these effects if they were due to self-demagnetization. It is shown here that the curves obtained by summing the magnetizations in all directions in a face-plane of an iron crystal, subject to no self-demagnetizing action,

contain no kinks. It therefore follows that, in this case, either no kinks exist with reference to any single direction of magnetization, or the kinks are so uniformly spread along the actual curve as to give it an appearance of strict continuity. Some method of obtaining unidirectional magnetization, uninfluenced by self-demagnetization, must therefore be devised before the real existence of kinks having that origin can be recognized.

Subsidiary Investigation. Dependence of Hysteresis on Field-strength.

It was decided to test this point experimentally. The crystal was maintained at 0° C. and magnetized through complete cycles beginning at low fields, in the manner already described, so that demagnetization was not effective. The hysteresis loss in these cycles was measured and plotted against the maximum field employed.

It is a well-known fact that hysteresis is small at low field strength and also at large field strength, so that there must be a maximum. Fig. 3 shows that this has not been attained, and should indicate an internal field of the order of perhaps 200 gauss.

I wish to express my thanks to the Carnegie Trust for the Scholarship held during the past two years, during which period this work was carried out.

LII. *Note on the Correlation of Time-series. By* Prof. J. T. Morrison, *University of Stellenbosch, South Africa**.

1. THE problem of the significance of high correlation coefficients between two variable quantities, whose values are given at successive equal intervals of time, is one which is of great importance in several branches of science—for instance, in meteorology, and in the study of industrial and economic movements, and of social changes in general. In many cases seasonal variations are eliminated by giving mean annual values,

* Communicated by the Author.

and these are extended for both variables over as long a period as possible.

In such cases if we draw the graphs of the variables against time it is obvious that the degree of significance to be attached to an observed high correlation coefficient between the variables will depend on the relative predominance of the following characters in the two graphs:—

(1) The general increase or decrease, as shown by the slope of the straight line of linear regression of each variable taken against time.

(2) The number, positions, and amplitudes of the arcs cut off by the line of regression from the smoothed line that shows the long-term trend of the variable.

(3) The short-term deviations from the long-term smoothed line.

Clearly if (1) is predominant relatively to (2) and (3) in two series of utterly unrelated quantities a high correlation coefficient will arise which will be completely without significance. This is essentially the cause of the "nonsense correlations" dealt with by Yule [(1)] in his presidential address to the Royal Statistical Society in 1926. If, however, a high correlation coefficient exists between two variable quantities in which features (2) and (3) are well-marked, and which are not unlikely to have a bearing one on the other, then the high correlation coefficient must be regarded as significant of a real correlation.

2. Fisher [(2)] has shown how any variable quantity whose values are given over a succession of n units of time can be represented by a linear sum of a set of normalized orthogonal polynomial functions of the time, of degrees 0, 1, 2, . . .$(n-1)$. In this case the long-term trend of the quantity is given, *for that period and for that period only*, by the sum of the early terms of the series. Fisher [(3)] points out that the coefficients by which the successive functions are multiplied in the terms of the series are proportional to the direction-cosines of a vector in n-dimensional space representative of the variable quantity, and that the correlation coefficient between two variables is equal to the cosine of the angle between their representative vectors.

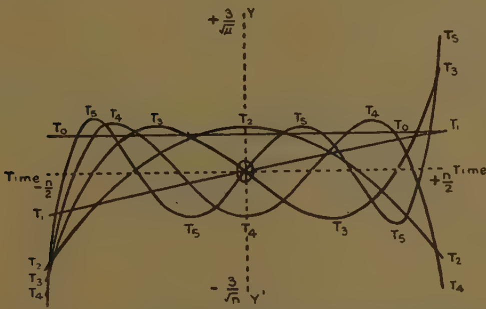
This is really the gist of the whole matter as regards likelihood of significance, but it appears to be worth while to go into the question somewhat more explicitly.

3. Fisher[(4)] reckons time from the midpoint of the period concerned, so that the n values of the variable considered, say $y_1, y_2, y_3, \ldots y_n$ belong to the instants of time

$$-\frac{n-1}{2},\ -\frac{n-3}{2},\ \ldots -\frac{3}{2},\ -\frac{1}{2},\ +\frac{1}{2},$$
$$+\frac{3}{2},\ \ldots +\frac{n-3}{2},\ +\frac{n-1}{2}.$$

The polynomial functions of the time, written by Fisher in the form $T_0, T_1, \ldots T_{n-1}$, are of degrees 0, 1, $(n-1)$, and are specified by the following defining equations :—

$$\mathop{S}_{t=-\frac{n-1}{2}}^{+\frac{n-1}{2}}(T_r^2)=1; \quad \mathop{S}_{t=-\frac{n-1}{2}}^{+\frac{n-1}{2}}(T_rT_s)=0,\ r\neq s. \quad . \quad \begin{cases}(3.1)\\(3.2)\end{cases}$$



It follows at once from these equations that, as n becomes large, the functions approach as limiting values the Legendre polynomials, of which the first six are

$$T_0=\frac{1}{\sqrt{n}}, \qquad T_1=2\sqrt{\frac{3}{n}}t,$$

$$T_2=\frac{6}{n^2}\sqrt{\frac{5}{n}}\left(t^2-\frac{n^2}{12}\right), \qquad T_3=\frac{20}{n^3}\sqrt{\frac{7}{n}}\left(t^3-\frac{3n^2}{20}t\right),$$

$$T_4=\frac{70}{n^4}\sqrt{\frac{7}{n}}\left(t^4-\frac{3n^2}{14}t^2+\frac{3n^4}{560}\right), \qquad T_5=\frac{252}{n^5}\sqrt{\frac{11}{n}}\left(t^5-\frac{5}{18}t^3+\frac{15t}{1008}\right).$$

The functions are, of course, independent of the variable y. They are defined for the n units of time only.

Graphs of $T_0 \ldots T_5$ are given in the figure. The graphs

show clearly the approximate uniformity of the amplitudes of the arcual parts of T_2 to T_5 and the rapid increase of the terminal parts—characteristics which have to be borne in mind in interpreting the results of the analysis of time-series by means of these functions.

4. If we now with Fisher [5] write

$$y=\alpha_0 T_0+\alpha_1 T_1+\alpha_2 T_2+\ldots \alpha_{n-1} T_{n-1}, \quad . \quad (4.1)$$

where $\alpha_0 \ldots \alpha_{n-1}$ are independent of n, we observe that the n given values of y taken with the values of $T_0 \ldots T_{n-1}$ at the corresponding instants of time furnish us with n linear equations to determine the coefficients $\alpha_0 \ldots \alpha_{n-1}$, so that the coefficients are uniquely determined. Each coefficient can, however, be at once calculated directly, for, if we form the sum of the n products yT_s, we have

$$\begin{aligned} \mathop{S}_{t=-\frac{n-1}{2}}^{+\frac{n-1}{2}}(yT_s) &= S(\alpha_0 T_0 T_s+\alpha_1 T_1 T_s+\ldots \alpha_s T_s^2+\ldots \alpha_{n-1} T_{n-1} T_s) \\ &= \alpha_s S(T_s^2)=\alpha_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2) \end{aligned}$$

Since

$$S(T_0^2)=1, \quad \therefore\ T_0=\frac{1}{\sqrt{n}}. \quad . \quad . \quad . \quad (4.3)$$

Also

$$T_0 S(T_s)=S(T_0 T_s)=0, \quad \therefore \mathop{S}_{t=-\frac{n-1}{2}}^{+\frac{n-1}{2}}(T_s)=0, \quad s\neq 0. \quad (4.4)$$

Again,

$$\frac{S(y)}{n}=\frac{S(\alpha_0 T_0)}{n}=\frac{\alpha_0}{\sqrt{n}}=\alpha T_0,$$

$$\therefore\ \alpha_0 T_0=\text{the arithmetic mean of the } ys=\bar{y}. \quad (4.5)$$

Also

$$\begin{aligned} \mathop{S}_{1}^{n}\{(y-\bar{y})^2\} &= S\{(\alpha_1 T_1+\alpha_2 T_2 \ldots \alpha_{n-1} T_{n-1})^2\} \\ &= \alpha_1^2+\alpha_2^2+\ldots \alpha_{n-1}^2=n\sigma_y^2. \quad . \quad . \quad . \quad (4.6) \end{aligned}$$

Hence the standard deviation

$$\sigma_y=\sqrt{\frac{\alpha_1^2+\ldots \alpha_{n-1}^2}{n}}. \quad . \quad . \quad . \quad . \quad (4.7)$$

5. The following regression coefficients may now be noted:—

(*a*) The linear regression coefficient of y on *time* is given by

$$r_{yt}=\frac{S(\alpha_1T_1+\ldots\alpha_{n-1}T_{n-1})t}{\sqrt{S\{(y-\bar{y})^2\}S(t^2)}}$$

$$=\frac{S(\alpha_1T_1^2+\alpha_2T_1T_2+\ldots\alpha_{n-1}T_1T_{n-1})}{\sqrt{n\sigma_y^2}\times S(T_1^2)}=\frac{\alpha_1}{\sqrt{n\sigma_y^2}}. \quad (5.1)$$

(*b*) The regression coefficient of y on the term α_sT_s, which we may write

$${}_sr_{yt}=\frac{S(\alpha_1\alpha_sT_1T_s+\ldots\alpha_s^2T_s^2+\ldots\alpha_{n-1}\alpha_sT_{n-1}T_s)}{\sqrt{n\sigma_y^2}\times S(\alpha_s^2T_s^2)}$$

$$=\frac{\alpha_s}{\sqrt{n\sigma_y^2}}. \quad (5.2)$$

(*c*) Similarly, if the long-term trend of the series be sufficiently represented by the group of terms $\alpha_1T_1+\alpha_2T_2+\ldots\alpha_rT_r$, it is found that the regression coefficient of y on this group is

$$\frac{\sqrt{\alpha_1^2+\alpha_2^2+\ldots\alpha_r^2}}{\sqrt{n\sigma_y^2}}={}_rR_{yt}\text{ (say)}. \quad (5.3)$$

Results (5.2) and (5.3) suggest that it would be well to regard

$$\frac{\alpha_1}{\sqrt{n\sigma_y^2}},\quad\frac{\alpha_2}{\sqrt{n\sigma_y^2}}\ldots\frac{\alpha_r}{\sqrt{n\sigma_y^2}}$$

as the fractions which specify the long-term trend of the time-series. We shall write them C_{y_1}, C_{y_2}, C_{yr}, and name them the long-term *characteristics* of the series.

Clearly

$${}_rR_{yt}=\sqrt{C_1^2+C_2^2+\ldots C_r^2}. \quad (5.4)$$

${}_rR_{yt}$ may be regarded as the measure of the *total amplitude* of the long-time trend.

6. It is easy to show analytically, as is indeed nearly obvious from Fisher's geometrical statement, that the linear polynomial expression $\alpha_1T_1+\alpha_2T_2+\ldots\alpha_rT_r$ is that polynomial of the rth degree, on which y has the highest possible regression coefficient, and which therefore leaves

the smallest mean-square residuals when subtracted from the *ys*.

For any other polynomial of the *r*th degree may be written

$$(\alpha_1+\delta_1)T_1+(\alpha_2+\delta_2)T_2+\dots(\alpha_r+\delta_r)T_r,$$

and the regression coefficient (say ${}_rR'_{yt}$) of y on it is

$$\frac{S\{(\alpha_1T_1+\dots\alpha_{n-1}T_{n-1})(\overline{\alpha_1+\delta_1}T_1+\dots\overline{\alpha_r+\delta_r}T_r)\}}{\sqrt{n\sigma_y^2(\alpha_1^2+\alpha_2^2\dots\alpha_r^2+2\alpha_1\delta_1+\dots2\alpha_r\delta_r+\delta_1^2+\dots\delta_r^2)}}$$

$$=\text{(say)}\ \frac{A^2+\overset{r}{\underset{1}{S}}\alpha\delta}{\sqrt{n\sigma_y^2(A^2+2\overset{r}{\underset{1}{S}}\alpha\delta+D^2)}},$$

where
$$A^2=\overset{r}{\underset{1}{S}}(\alpha^2),\qquad D^2=\overset{r}{\underset{1}{S}}(\delta^2),$$

$$\therefore\quad \frac{{}_rR'^2_{yt}}{{}_rR^2_{yt}}=\frac{A^4+2A^2\overset{r}{\underset{1}{S}}\alpha\delta+(\overset{r}{\underset{1}{S}}\alpha\delta)^2}{(A^2+2\overset{r}{\underset{1}{S}}\alpha\delta+D^2)\times A^2}.$$

But
$$A^2D^2>(\overset{r}{\underset{1}{S}}\alpha\delta)^2,\qquad\therefore\quad {}_rR'^2_{yt}<{}_rR^2_{yt}.\qquad (6.1)$$

7. Correlation of two time-series.

We can now express the regression coefficient of y on any other time-series z, given also for n units of time, in terms of the sets of characteristics of the two series,

$$C_{y_1}\dots C_{y(n-1)}\quad\text{and}\quad C_{z_1}\dots C_{z(n-1)}.$$

For since
$$(y-\bar{y})=\alpha_{y_1}T_1+\dots+\alpha_{y(n-1)}T_{n-1},$$
and
$$(z-\bar{z})=\alpha_{z_1}T_1+\dots+\alpha_{z(n-1)}T_{n-1},$$

$\therefore$ the regression coefficient

$$r_{yz}=\frac{\overset{n-1}{\underset{1}{S}}\{(y-\bar{y})(z-\bar{z})\}}{\sqrt{n\sigma_y^2\times n\sigma_z^2}}=\frac{\alpha_{y_1}\alpha_{z_1}+\dots\alpha_{y\overline{n-1}}\alpha_{z\overline{n-1}}}{\sqrt{n\sigma_y^2\times n\sigma_z^2}}$$

$$=\frac{\alpha_{y_1}}{\sqrt{n\sigma_y^2}}\times\frac{\alpha_{z_1}}{\sqrt{n\sigma_z^2}}+\dots\frac{\alpha_{y\overline{n-1}}}{\sqrt{n\sigma_y^2}}\times\frac{\alpha_{z\overline{n-1}}}{\sqrt{n\sigma_z^2}}$$

$$=C_{y_1}C_{z_1}+\dots+C_{y_{n-1}}C_{z_{n-1}}.\qquad (7.1)$$

If only the first r characteristics are moderately large fractions in the two series, so that these are sufficient to characterize the long-term trends, then the portion of the total regression coefficient that arises from the long-term trend is

$$C_{y_1}C_{z_1}+\ldots+C_{yr}C_{zr}={}_rR_{yz}. \quad . \quad . \quad . \quad (7.2)$$

We thus disentangle that part of the regression-coefficient of y on z which arises from the long-term trends of the two, and are able to form some estimate of the significance of that part of the coefficient.

The significance of ${}_rR_{yz}$ is to be estimated not so much by its absolute magnitude as by its relation to the *amplitudes* of the two long-term trends defined in 5·4. The significance depends on

$$\frac{C_{y_1}C_z+\ldots C_{yr}C_{zr}}{\sqrt{(C^2_{y_1}+C^2_{y_2}+C^2_{yr})(C^2_{z_1}+C^2_{z_2}+\ldots+C^2_{zr})}}. \quad . \quad (7.3)$$

For given numerical values of the characteristics, C_y and C_z, the fraction (7·3) will attain its greatest numerical value when all the terms of the numerator are of the same sign. For a given value of the denominator this *maximum value* of the fraction is determined by the squares of the differences of the numerical values of corresponding y and z characteristics. For

$$\frac{|{}_yC_1|\,|{}_zC_1|+|{}_yC_r|\,|{}_zC_r|}{\sqrt{(C^2_{y1}+\ldots+C^2_{yr})(C^2_{z_1}+\ldots C^2_{zr})}}$$

$$=\frac{1}{2}\frac{(C^2_{y_1}+\ldots+C^2_{yr})+(C^2_{z_1}+\ldots+C^2_{zr})-\{(|C_{y_1}|-|C_{z_1}|)^2+\ldots+(|C_{yr}|-|C_{zr}|)^2\}}{\sqrt{(C^2_{y_1}+\ldots+C^2_{yr})(C^2_{z_1}+\ldots+C^2_{zr})}}$$

$$. \quad . \quad . \quad . \quad (7.4)$$

The *maximum value* of the fraction will be unity even if

$$\overset{r}{\underset{1}{S}}C^2_{ys}\neq\overset{r}{\underset{1}{S}}C^2_{zs},$$

provided

$$\frac{C_{y_1}}{C_{z_1}}=\frac{C_{y_2}}{C_{z_2}}=\ldots\frac{C_{yr}}{C_{zr}}=k \text{ (say)}. \quad . \quad . \quad (7.5)$$

It will approach unity if C_{y_1} and C_{z_1} are both large relatively to the remaining characteristics. In such a case a high regression-coefficient cannot be regarded

as significant. If, however, several of the coefficients are considerable, a high regression-coefficient would be improbable in the absence of a true correlation, and the existence of such a coefficient must be regarded as significant. That is obvious from equations 7·4 and 7·5. It is to be noted that $\overset{n-1}{\underset{1}{\mathrm{S}}}(\mathrm{C}^2)=1$, and therefore $\overset{r}{\underset{1}{\mathrm{S}}}(\mathrm{C}^2)<1$, and that therefore the presence of a large C_{y_1} and a large C_{z_1} must make the remaining characteristics small. When, however, the variables make well-marked fluctuations during the specified period it will become improbable that the fluctuations will fit, and therefore the value of $\underset{1}{\mathrm{S}}\{(\,|\,\mathrm{C}_{ys}\,|-|\,\mathrm{C}_{zs}\,|\,)^2\}$ is likely to be large. In such cases an observed close fit with consequent high correlation coefficient must be regarded as highly significant.

8. It cannot be too strongly emphasized that the linear sum of the first r polynomials which gives the long-term trend of a variable quantity for n units of time cannot be used to predict the probable course of that variable at the end of that period. The whole analysis depends on the properties that for the n units of time $\mathrm{S}(\mathrm{T}_r\mathrm{T}_s)=0$, $r\neq s$, and $\mathrm{S}(\mathrm{T}_s{}^2)=1$, and neither property holds good for any extension of the period. The coefficients of the terms of each constituent polynomial are all functions of n, and all the zeros of the polynomials fall within the period, so that beyond its limits they all diverge rapidly from their mean value within the period. They simply furnish us therefore with a mathematical method of expressing the various bendings of the variable within the specified period, and have no significance beyond it.

It does not, however, follow that the significance of an observed high correlation coefficient between the long-term trends of two variables also ceases beyond the observed period. The position rather appears to be that the analysis does not allow us to predict the movement of either variable, but that it does permit us to estimate the probability of the movement of one variable when that of the other is known.

9. Seasonal variations, *i. e.*, variations that occur in course of a year, estimated by taking mean values for a considerable number of years, do not appear to present

suitable material for analysis by polynomials. The values of the variable in this case and its rates of change are necessarily approximately the same at the end of the year as at the beginning, and this is far from being true of the orthogonal polynomials. In the case of each polynomial of even degree the terminal values are indeed equal, but the values of their rates of change are equal and opposite; while in the polynomials of odd degree the terminal rates of change are identical but the values equal and opposite. Further, near the boundaries of the period the values and rates of change of the polynomials depart rapidly from the mean values and mean rates of change.

In this case an ordinary Fourier analysis, using normalized simple harmonic functions, provides a tool having all the properties discussed in the early parts of this note.

10. When the values of the long-term trends of two series have been deducted from the actual values the correlation of the short-term residuals falls to be examined, with positive, zero, or negative lagging of the one set over the other. With zero lagging these residuals contribute to the regression coefficient of the whole series the amount $r_{yz} - {}_rR_{yz}$; but this is smaller than their own mutual regression coefficient with zero lagging in the ratio of the product of their standard deviations to the product of the standard deviations of the two whole series. The residuals are usually assumed to be normally distributed, and the ordinary test is therefore used in estimating the significance of the various regression coefficients obtained with varying amounts of lagging. The information obtained in this way is often of great importance in practice.

11. *Summary.*

In studying the correlation of two time-series a preliminary analysis of each by polynomials or by simple harmonic functions allows us (1) to state with precision the characteristics of their long-term trends, (2) to write down at once that portion of their regression coefficient which arises from these trends, (3) to make some estimate of the significance of that coefficient, (4) to deal separately with the correlation of short-term residuals, with or without various degrees of lagging.

It is pointed out that the polynomial expression for the long-term trend of a series cannot be used for prediction beyond the period discussed, and also that for seasonal variations simple harmonic functions and not polynomials are the proper constituent functions to be used in the analysis.

12. *References.*

(1) Yule, J. S. S. lxxxix. pp. 1 *et seq.*
(2) R. A. Fisher, Phil. Trans. series B, ccxiii. pp. 89–142.
(3) R. A. Fisher, *loc. cit.* p. 100.
(4) R. A. Fisher, *loc. cit.* p. 99.
(5) R. A. Fisher, *loc. cit.* p. 97.

LIII. *The Thermal Properties of Benzene-Air and Methyl Alcohol-Air Mixtures.*

By James Small, *B.Sc., Ph.D,. Carnegie Teaching Fellow, The James Watt Engineering Laboratories, The University of Glasgow* *.

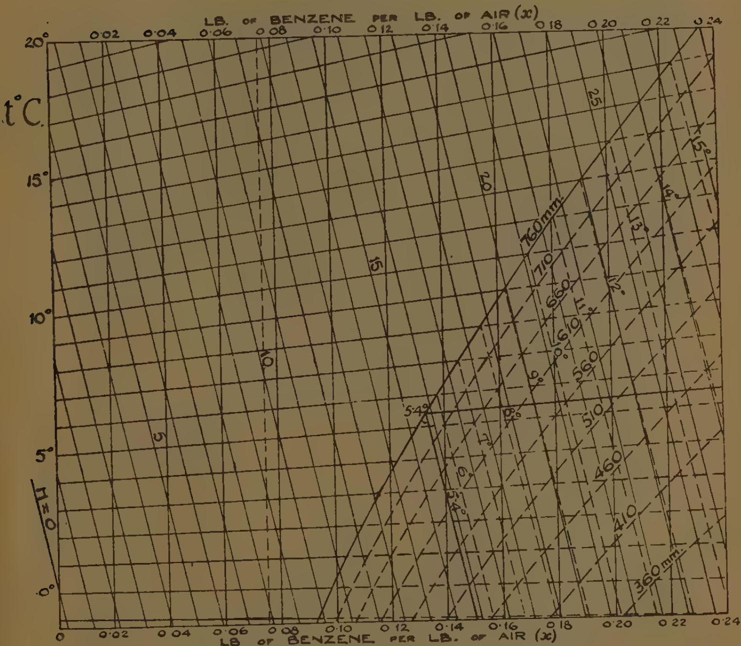
In a previous contribution † the writer dealt with the characteristics of a mixture of air and ethyl-alcohol, and showed by means of charts how the total heat of the constituents before mixing and the total pressure of the subsequent mixture control its thermal state.

As a supplement to the analysis there given charts for a mixture of benzene and air and for a mixture of methyl-alcohol and air have been prepared, which afford material for drawing a comparison between the properties of the different binary mixtures with reference to the nature of the volatile constituents.

Benzene-Air Mixtures.—Benzene freezes at 5·4° C., but in preparing Table I., which contains the thermal properties of benzene, it has been assumed that it remains liquid down to 0° C. The values of sensible and of total heat for a mass of 1 lb. have been reckoned from a datum state which is imaginary, namely, liquid benzene at 0° C. The sensible heat is taken as independent of the pressure. So long as Table I. and the chart in fig. 1, constructed with its aid, are not applied to conditions in which freezing would actually take place no error arises.

* Communicated by the Author.
† Phil. Mag. ser. 7, xvi. p. 641 (Sept. 1933).

Fig. 1.



THERMAL CHART FOR BENZENE-AIR MIXTURES

TABLE I.

The Thermal Properties of Benzene.

t °C.	Liquid specific heat (mean).	Liquid or sensible heat (cals./gr. or C.H.U./lb.),	Latent heat of vaporization (cals./gr. or C.H.U./lb.).	Total heat of dry-saturated vapour (cals./gr. or C.H.U./lb.).
0	0·385	0	107·0	107·0
20	0·395	7·9	103·8	111·8
40	0·404	16·2	100·7	116·9
60	0·414	24·8	97·5	122·3
80	0·424	33·9	94·2	128·1
100	0·434	43·4	90·6	134·0
120	0·444	53·3	86·7	140·0
140	0·454	63·6	82·6	146·2
160	0·464	74·2	78·5	152·7

The values are abstracted from the information collected in the International Critical Tables (vol. v.).

It is found that within the range of partial pressures experienced in carburation the total heat of the vapour, whether dry-saturated or unsaturated, is almost independent of pressure, and is very approximately expressible in terms of the temperature by $H_f = 107 + 0{\cdot}27t$. The total heat of a mixture of 1 lb. of air and x lb. of unsaturated benzene vapour is therefore given by $H_m = 0{\cdot}24t + x(107 + 0{\cdot}27t)$, where 0·24 is the specific heat of the air at constant pressure, $t°$ C. is the temperature, and H_m is in calories per gramme or lb.-degree C. heat units (C.H.U.) per lb.

TABLE II.

Vapour Pressures of Benzene *.

$t°$ C.	p_f' mm.	$t°$ C.	p_f' mm.
0	26·6	12	50·2
2	29·7	14	55·6
4	33·1	16	61·4
6	36·8	18	67·8
8	40·9	20	74·3
10	45·2	22	80·9

If P is the total pressure of an unsaturated mixture, made up of the partial pressures p_a and p_f of the 1 lb. of air and x lb. of benzene respectively, then $x = \frac{M_f}{M_a} \cdot \frac{p_f}{P - p_f}$, where $P - p_f = p_a$ and M_a and M_f are the respective molecular weights. For benzene, $M_f = 78{\cdot}05$; and $M_a = 28{\cdot}97$. Hence the quantity of benzene, x' lb., which saturates the mixture at t° C. is $x' = \frac{78{\cdot}05}{28{\cdot}97} \cdot \frac{p'_f}{P - p'_f}$, where p'_f is the saturation pressure of benzene at $t°$ C. Or, evaluating the coefficient, $x' = 2{\cdot}694 \frac{p'_f}{P - p'_f}$.

The values of p'_f are given in Table II. for the range of temperatures in which we are interested, and from these, following the method of construction described in the

* Based on the figures of Young, Sci. Proc. Roy. Dublin Soc. xii. no. 31 (June 1910).

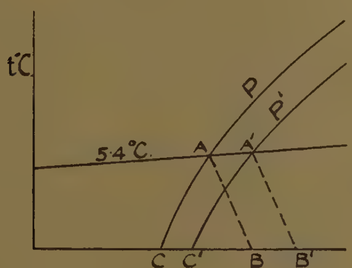
article to which reference has already been made *, the chart in fig. 1 is prepared.

A number of saturation lines corresponding to different total pressures are drawn which render it possible to study the conditions of the mixture at total pressures below 760 mm.

Lines of constant temperature have been drawn in the saturation region—that is, the region below and to the right of a saturation curve—but these apply only to a total pressure of 760 mm. For practical purposes such lines may be regarded as parallel to the lines of total heat.

A line of constant temperature in the dry region meets the saturation curve corresponding to a total pressure P (fig. 2) at the point A. The constant temperature line

Fig. 2.



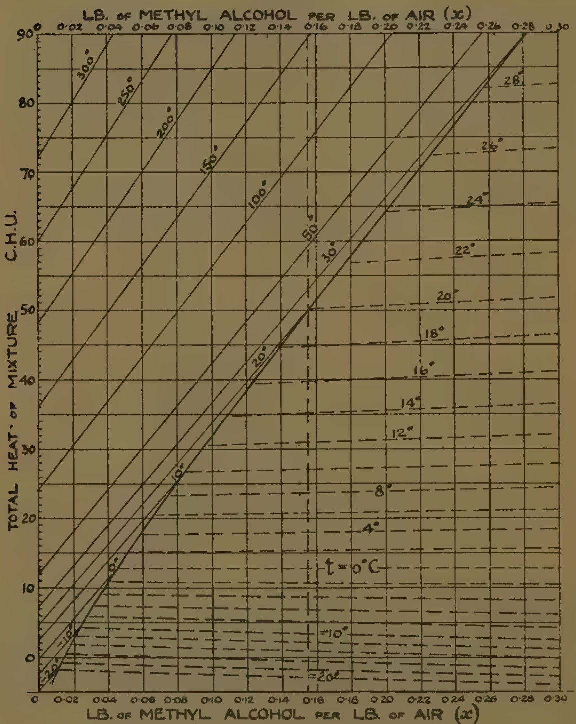
in the region of saturation is AB ; but if the total pressure is P′, the line now cuts the saturation curve in A′, and in the region of saturation the line is A′B′. If the temperature in question is 5·4° C., the freezing-point of benzene, then readings of the chart within such areas as ABC, A′B′C′ (according as the total pressure is P or P′) are not valid.

The chemically correct weight of air to burn 1 lb. of benzene is 13·25 lb.—that is to say, the chemically correct quantity of benzene for combustion in 1 lb. of air is 0·0754 lb. The vertical dotted line in fig. 1 drawn at this value of x shows that at all working temperatures the mixture is dry.

* Phil. Mag. ser. 7, xvi. p. 641 (Sept. 1933).

The total heat of 1 lb. of air at 15° C. is 3·6 C.H.U., and of 0·0754 lb. of liquid benzene at 15° C. (see Table I.) is 0·44 C.H.U. The total heat available in the constituents is therefore only 4·04 C.H.U. For $x=0·0754$ this is

Fig. 3.



THERMAL CHART FOR METHYL ALCOHOL-AIR MIXTURES

quite below the range of the chart. If it is assumed, however, that carburetter temperatures below 0° C. are not desirable, then, according to the chart, the total heat should be no less than 8·2 C.H.U. Thus a quantity of heat amounting to over 4 C.H.U. is to be applied either by pre-heating the air or by heating the mixture. If the

former, the air must enter the carburetter at a temperature over 32° C.

Methyl Alcohol-Air Mixtures.—The properties of methyl alcohol are not so amenable to this graphical treatment as those of water-vapour, ethyl alcohol, and benzene. The figures obtainable tend to show that the total heat of the dry vapour varies considerably with pressure, so that the values are not approximately expressible simply in terms of the temperature, at least over any considerable range.

TABLE III.

The Thermal Properties of Methyl Alcohol.

t° C.	Liquid specific heat (mean).	Liquid or sensible heat (cals./gr. or C.H.U./lb.).	Latent heat of vaporization (cals./gr. or C.H.U./lb.).	Total heat of dry-saturated vapour H_s (cal./gr. or C.H.U./lb.).
−40	0·536	−21·4	294·8	273·4
−30	0·548	−16·4	292·2	275·8
−20	0·552	−11·0	289·8	278·8
−10	0·561	− 5·6	287·0	281·4
0	0·566	0	284·5	284·5
10	0·575	5·7	281·5	287·2
20	0·585	11·7	278·9	290·6
30	0·590	17·7	275·9	293·6
40	0·595	24·0	272·3	296·3
50	0·598	29·9	268·8	298·7
60	0·601	36·1	265·2	301·3
70	0·603	42·2	260·5	302·7
80	0·605	48·4	253·2	301·6

If H_s is the total heat of 1 lb. of dry saturated vapour at temperature t', then the total heat of 1 lb. of superheated vapour at temperature t, and the same pressure is equal approximately to $H_s+0{\cdot}4(t-t')$. Thus the total heat of a mixture of 1 lb. of air and x lb. of vaporized methyl alcohol at t is therefore $H_m=0{\cdot}24t+x(H_s+0{\cdot}4(t-t'))$.

Over the range covered by the chart in fig. 3 it is found that H_m is nearly a linear function of x when t is constant.

The properties of methyl alcohol given in Table III. are based on figures of different authorities given in the International Critical Tables (vol. v.), and the vapour

pressures given in Table IV. are taken from the same source (vol. iii.). The values of sensible and total heat are reckoned from a datum of 1 lb. of liquid at 0° C.

The molecular weight of methyl alcohol is 32·03, and x' is therefore equal to $\frac{32\cdot03}{28\cdot97}\frac{p'_f}{P-p'}$ or $x'=1\cdot1056\ \frac{p'_f}{P-p'_f}$. The chart in fig. 3 has been constructed on a different plan from that in fig. 1. It represents an alternative construction which can be used in the case of other volatiles. Lines of constant total heat are horizontal, and the saturation curve is constructed for any given total pressure P by choosing certain temperature values, using the corresponding values of p'_f to find x', and hence

Table IV.

Vapour Pressures of Methyl Alcohol.

t° C.	p'_f mm.	t° C.	p'_f mm.
−20	7·2	30	157·0
−10	15·6	40	255·0
0	29·6	50	405·5
10	53·8	60	619·4
20	93·9	70	929·0

determining the values of H_m. This method lends itself specially to the case where the total heat of a vapour is not satisfactorily expressible by a linear function of t, and it is not feasible to begin by drawing a series of sloping temperature lines.

In fig. 3 only the saturation curve for a total pressure of 760 mm. is drawn.

The chemically correct value of x for combustion is 0·155, the vertical dotted line being drawn at this value. The total heat and temperature of the mixture for complete vaporization of this proportion of fuel are 50 C.H.U. and 20° C. respectively. If the liquid fuel and the air are initially at 15° C. the quantity of heat to be supplied to bring about complete vaporization is therefore 45 C.H.U., and this must be done by preheating the air and by heating the mixture. If it were supplied by preheating the air alone the air temperature required would be the extremely high value of 200° C.

LIV. *The Blackness of a Black-body Radiator.*

By F. A. Cunnold, *B.A.*, *and* M. Milford, *M.A.**

THE investigation of the emissivity of various metals at high temperatures has often made it necessary to realise a "black-body" radiator whose walls are composed of the metal investigated. By means of an optical pyrometer sighted on the wall and on the cavity alternately, the true temperature and the brightness temperature of the same piece of metal can be obtained and the emissivity calculated.

Several forms of black-body have been used for this purpose, notably the V-shaped wedge [(1)], the helical filament [(2), (3), (4)] and the continuous tube pierced by a small hole[(5)].

Any theoretical investigation of the blackness must take into account the radiation from every element of the inner surface of the cavity which may eventually emerge and reach the measuring instrument. If the radiator is assumed to be perfectly reflecting, it is possible to find the positions of the relevant elements of surface by the geometrical construction of a ray, considered as retracing its path. This can be done (for the tube) in any plane perpendicular to the axis [(6), (7)] and also, with somewhat greater complication, in other planes. In this way the areas of the elements and the number of reflexions the radiation from each element undergoes before emerging could be obtained, and hence, if the summation (over the aperture used and over the range of angles included by the entrance-cone of the pyrometer) proved possible, the total energy entering the measuring instrument could be calculated. By comparison with the black-body formula for the temperature of the enclosure, this process would give a theoretical method of assessing the blackness, but it should be clear, even from this short summary, that the problem is too complicated for solution without assumptions which would seriously impair the validity of the answer.

The process is further confused by the fact that, in practice, the inner surface of the tube is by no means

* Communicated by A. Egerton, F.R.S.

perfectly reflecting, and there are two other sources of error which may be mentioned here :—

(1) Diffusion of radiation towards the cold ends of the tube, and

(2) Errors due to the temperature gradient along the tube.

Worthing [5] concluded that these two effects were negligibly small, and there seems every reason to believe that this is also true of the tubes used in the present work.

In the same connexion it is of interest to note that both Worthing and Ives [5], [6] investigated the temperature distribution in the plane through the hole perpendicular to the axis of the tube, and concluded that there was no observable lack of uniformity.

In the present case a black-body was required for the calibration of a precision optical pyrometer which could be conveniently maintained at standard temperatures and in which melting-points could be observed. The method has been described elsewhere [8], but it may be remarked that for this work, since the temperature is never higher than 1800° K., and seldom above the gold-point (1336° K.), platinum tubes are very suitable. As they have to be thin, such tubes are not costly.

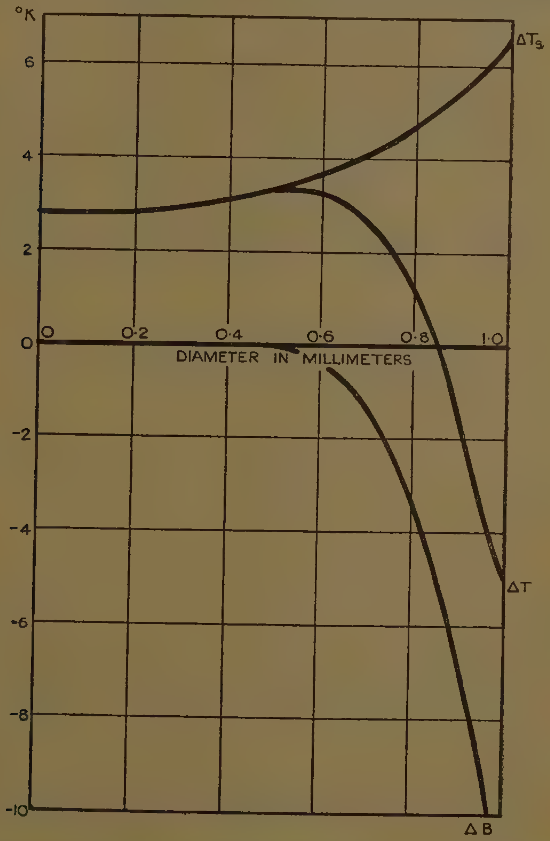
It is clearly of great importance to assess the blackness of the radiation from the small hole drilled in the wall of the tube (which is used to give the true temperature of the cavity), and in view of the complication of the theoretical treatment it was decided to attempt to assess the blackness experimentally.

The method which suggests itself is that of comparing the radiation from two holes, one of which (no. 1) is kept constant in size, as a standard, while the other (no. 2) is enlarged progressively. This can be simply done by means of the optical pyrometer, when the differences will be expressed in terms of the apparent differences in temperature. If the difference is measured for various sizes of hole, it should be possible to extrapolate to zero size of hole, and thus to obtain the correction for any known hole.

Shackelford [9] investigated the effect of the spacing of the turns on the brightness ratio of the radiation from the inside and outside of a helical filament, and extrapolated to zero spacing in a somewhat similar manner, but,

so far as we have been able to discover, the method has never been applied to tubes.

As might have been expected, however, the change of apparent temperature of the hole is also affected by the fact that enlargement of the hole decreases the cross-



section of the tube available for carrying the current and causes a local rise in temperature.

If the radiation temperatures of the two holes are T_1 and T_2, the differences ΔT $(=T_2-T_1)$ must be due to two factors: (1) the difference of blackness which it is desired to assess, and which may be expressed in tempera-

ture units as ΔB, and (2) the difference of the true temperatures $\Delta T_s = (T_{s_2} - T_{s_1})$ of the metal at the holes, where T_{s_1} and T_{s_2} are the true temperatures at the two holes respectively.

That is,

$$\Delta T = \Delta B + \Delta T_s$$

or

$$\Delta B = \Delta T - \Delta T_s.$$

Now, T_1 and T_2 are the temperatures of the holes as they are measured directly with the pyrometer, and hence ΔT is known. Further, the true temperatures of the metal can be obtained from the apparent (brightness) temperatures S_1 and S_2 of the surface near each hole, if we assume that the emissivity of the surface is constant for small changes of temperature. For if the emissivity is e_λ,

$$\log e_\lambda = \frac{C_2}{\lambda}\left[\frac{1}{S} - \frac{1}{T_s}\right],$$

or, if e_λ is constant,

$$\frac{1}{S} - \frac{1}{T_s} = \text{const.}$$

and

$$\frac{\Delta S}{S^2} = \frac{\Delta T_s}{T_s^{\,2}},$$

or

$$\Delta T_s = \left(\frac{T_s}{S}\right)^2 \Delta S.$$

Now T_s/S will be very nearly equal to T/S, since the difference between the true temperature of the surface and the radiation temperature of the hole is never more than 10° and T is of the order of 1300° K.; we can accordingly take T/S for T_s/S in calculating ΔT_s from the observed ΔS.

The procedure is, therefore :—At each size of hole, measure ΔT and ΔS ; calculate ΔT_s, when $\Delta T - \Delta T_s$ gives the blackness correction.

In practice the method adopted was as follows. The tube was set up with the holes placed centrally and the current adjusted so that the temperature, as measured by the standard hole, was as nearly as possible at the gold-point. When the temperature seemed steady, observations were made on the two holes alternately. To observe

the surface temperature it was found necessary to turn the tube round, so that the holes appeared well to one side; otherwise it was very difficult to match the pyrometer filament on the surface in the immediate neighbourhood of the hole, since the bright circle seemed to distract the attention from the small parts on each side where the matching had to be done. If the tube cannot be turned *in situ*, it is still sufficiently accurate if the temperature is reset to within 10° at the gold-point, since only the differences between the two holes are observed. It is also unnecessary for the pyrometer to be calibrated absolutely, so long as it can be relied upon to give the differences correctly, and the method can, therefore, be applied to correct the calibration of a pyrometer which has been standardized at the gold-point on a platinum tube black-body of this type.

The curves show the variation of ΔT and ΔT_s with diameter of hole for a tube of internal diameter 1·7 mm. and 10 cm long between the holders, and the observations were made using a colour screen of effective wave-length 0·690 μ. It will be noticed that ΔT_s is never less than a finite value (about 2·7°); this is due to the temperature gradient in the tube—the holes were 4 mm. apart along the axis of the tube.

ΔT rises to a maximum at a diameter of about 0·55 mm. and then falls rapidly. ΔB, being the difference between the ordinates of the two curves, varies as shown. Even if the absolute positions of the curves ΔT and ΔT_s are relatively uncertain, the ΔB curve must be substantially correct, since the ΔT, ΔT_s curves cannot cross (for this would mean that $\Delta B > 0$), and even errors as high as 1°, above a diameter of 0·5 mm., will not appreciably affect the value of ΔB below this point.

It will be seen that since $\Delta B = -1°$ corresponds to a blackness of 99·9 per cent., the hole is black to this accuracy with a diameter as large as 0·65 mm. The correction for a hole of 0·3 mm., which is quite a convenient size, is clearly negligibly small.

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(2) Langmuir, Phys. Rev. vi. p. 138 (1915).
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LV. *A Valve-maintained Stretched-wire Vibrator.* *By* W. S. Stuart, *M.Eng., A.M.I.E.E.**

Summary.

An arrangement for producing sustained vibrations of a stretched wire by means of thermionic valves is given.

The complete apparatus serves as a very convenient generator of audio-frequency currents for laboratory purposes.

The vibrating wire frequency meter, in which the current whose frequency is to be determined is passed through a stretched wire passing between magnet poles, has been used by a number of experimenters in audio-frequency work. Owing to the low damping of such an arrangement the resonant point is strongly marked, resulting in a high degree of accuracy, and, in addition, by varying the distance between the wire supports a frequency calibration is easily applied.

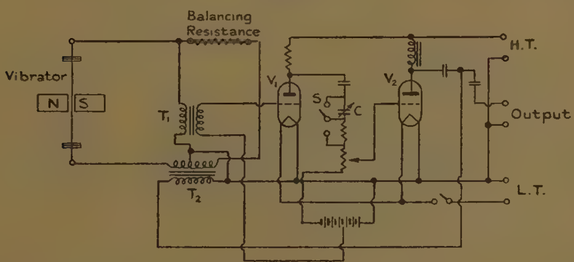
In the following description a method for producing sustained oscillations of the wire is given, the complete apparatus constituting a generator of audio-frequency currents which has been found extremely useful in the laboratory. Briefly, the arrangement consists of a back-coupled amplifier in which the wire vibrator replaces the usual parallel-tuned circuit. The frequency of operation is therefore the natural frequency of vibration of the wire, which may be varied by altering its length, but which, when set, is free from the drift commonly found in valve oscillators. The amplitude of the oscillations is governed by the degree of amplification provided by the valves, since the mechanical damping of the wire

* Communicated by the Author.

increases with its deflexion. The amplitude is therefore not controlled by the linearity of the valve characteristics, and in consequence the output has a very good wave form.

The circuit used by the author is shown in fig. 1 The vibrator is connected to the amplifier by a bridge circuit of which two arms are formed by the centre-tapped secondary of the current-transformer T_2, while the balancing resistance is adjusted to have a value equal to the ohmic resistance of the vibrating wire. In this way voltages applied to the grid of V_1 are due only to the movement of the wire and spurious oscillations of the amplifier are avoided. The transformer T_1 may have a high step-up ratio, and a value of 50 : 1 has been used

Fig. 1.



in the present case. The ratio of T_2 is chosen to match the impedance of the vibrator circuit to that of the output valve V_2, and a total value of 12·5 : 1 is used in the author's apparatus.

Phosphor bronze has been found to be the most satisfactory material for the stretched wire, which is no. 30 S.W.G. and about 1 metre in length. It is loaded with a weight of 6 lbs. by means of a pulley at one end, and its resistance is 0·4 ohm. With this arrangement by varying the distance between the knife edges a range of frequencies extending from 100 to 4000 cycles per second is easily obtained. The field-magnet, which is usually arranged to be in the middle of the vibrating portion of the wire, has poles 2 cm. square and produces a flux density of 6000 lines per sq. cm. in the air-gap.

It has been found necessary to employ two valves in the amplifier, with resistance capacity coupling as shown,

the voltage magnification being controlled by the potentiometer. The variable condenser in the coupling circuit, normally short-circuited by the switch S, is used to produce harmonics of the fundamental oscillations, and is referred to later.

Of the electrical power supplied to the vibrating wire a portion is used to maintain the oscillations, the remainder being dissipated in the wire resistance.

If we let r'=effective mechanical resistance to motion for a given deflexion, and v=effective velocity of centre point of the wire, such that the power dissipated due to the motion is

$$P_0=r'v^2,$$

then, if m=effective mass of the wire, the damping constant

$$\Delta=\frac{r'}{2m},$$

and substituting for r',

$$P_0=2m\Delta v^2.$$

Now let e=effective value of the voltage applied to the wire to balance its generated e.m.f., and i=wire current, then the motional impedance of the wire is

$$Z_m=e/i,$$

and the electrical power required to maintain the oscillations is

$$P_e=ei=\frac{e^2}{Z_m}=\frac{kv^2}{Z_m}.$$

The constant k relates e to v, and is therefore a measure of the variation of magnetic linkages for unit displacement of the wire.

We then have, since $P_0=P_e$,

$$2m\Delta v^2=\frac{kv^2}{Z_m},$$

or

$$\Delta=\frac{k}{2mZ_m}.$$

It is convenient to let the symbol M denote the voltage magnification obtained with the amplifier, that is, the e.m.f. transferred to the anode circuit of V_2 by its grid is M times as great as the e.m.f. actually induced in the wire. The ratio of the output transformer may be

represented by n, and we then have the total e.m.f. acting in the anode circuit of V_2 as

$$e(M-n).$$

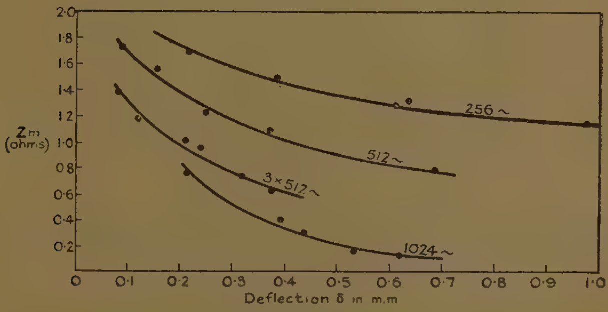
The wire current is accordingly

$$i=n\frac{e(M-n)}{\rho+2rn^2},$$

where ρ is the a.c. resistance of V_2 and r is the resistance of the wire, losses in the transformers being neglected. Then

$$Z_m=\frac{e}{i}=\frac{\rho+2rn^2}{n(M-n)},$$

Fig. 2.



from which the value of M required to maintain oscillations is given by

$$M=\frac{\rho+2rn^2}{nZ_m}+n.$$

The variation of Z_m with deflexion δ has been measured for several frequencies, the results being shown in fig. 2. The measurement was made by including the vibrator in a bridge circuit with three other resistances, as shown in fig. 3, the whole being connected to the amplifier in place of the wire alone. The measurement of δ is conveniently carried out by optical projection.

Since Z_m decreases with increasing values of δ it is convenient to write

$$\frac{1}{Z_m}=f(\delta),$$

so that the value of M given above becomes

$$M = f(\delta)\left(\frac{\rho}{n} + 2rn\right) + n.$$

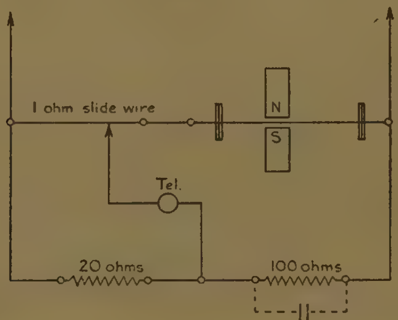
For a given value of δ, M is a minimum if the ratio of T_2 is such that

$$n^2 = \frac{\rho}{Z_m + 2r}.$$

Obviously a value of n to suit average conditions should be chosen.

It is often convenient, especially when the lower frequencies are in use, to produce harmonic vibrations

Fig. 3.



of the wire for the purpose of generating higher frequencies which are integral multiples of the fundamental. It has been found possible to do this by inserting in the intervalve coupling circuit a variable condenser C, whose maximum capacity is 0·0005 microfarad. This is brought into circuit by the switch S, which simultaneously cuts out a resistance in order to increase the total magnification available. Under these conditions the quantity M increases with frequency, and we therefore have a condition favourable to the production of harmonics. In the case of frequencies which are even multiples of the fundamental it is usually necessary to move the field-magnet some distance away from its central position between the knife edges. The use of the condenser introduces into the quantity M a quadrature component, so that it is no longer a wholly real number, as has hitherto been assumed.

In consequence the motional impedance Z_m contains a reactive component, and a condenser is introduced into the bridge circuit of fig. 3 when measuring this quantity. In fig. 2 the values of Z_m for the case of the curve relating to the third harmonic of the frequency 512 are therefore scalar values. They contain a negative reactance component, and the average vector slope is about 30 degrees.

Due to the complex nature of M there is a theoretical departure from the true resonant frequency of the vibrator, but in practice the low damping of the system reduces any error due to this to a negligible amount.

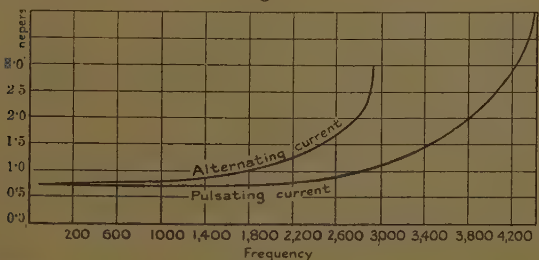
A voltage output from the amplifier is obtained from the anode of V_2, as shown in fig. 2. As the power at this point is limited to a comparatively small amount it is usually desirable to employ an additional power stage of capacity sufficient for the work in hand.

LVI. *Pulsating Currents Telephony.* By MARIO MARRO, *Dr.Ing.**

[Plates IX. & X.]

FOLLOWING my experimental research on a loaded telephone cable† I recorded the attenuation diagrams represented in fig. 1.

Fig. 1.



If, for instance, an alternating current of sinusoidal shape enters into a loaded telephone cable (Pupin system), and if the frequency of this current varies, the attenuation

* Communicated by the Author.

† "Alternating and Pulsating Currents in a Loaded Telephone Cable," Phil. Mag. xviii. p. 288 (1934).

of this current, recorded at the far end of the cable, depends on its frequency. There will be a limit frequency, behind which the attenuation for the alternating current will become infinite. This frequency is the cut-off frequency of the loaded cable—that is, the resonance frequency of the arrangement comprising the self coils put in series with the cable conductors and the cable capacity shunting the self coils. The loading sections of the cable, loaded by the Pupin system, act as a choke filter.

If I send into the loaded cable the same sinusoidal alternating current, which I rendered pulsating by suppressing one of the two half-waves by means of a rectifier, the attenuation curve will rise less quickly than that of the pure alternating current; the cut-off frequency has now a higher value. This fact will probably happen because the loading sections of the cable will correctly act as a choke filter in the case of steady state of the

Fig. 2.

alternating current, but not in transient condition, as in the case of half-waves spaced at one-half a cycle. These pulse currents are like those represented in fig. 2 (*a*).

On the contrary, let the alternating current be inserted in a circuit, including the rectifier, and the output be placed in the grid circuit of a valve whose anode circuit feeds the line (fig. 3). The valve acts as a buffer, so that the line will be terminated correctly.

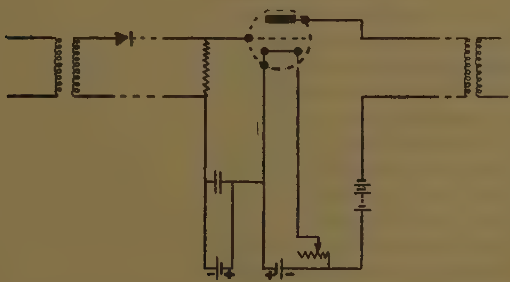
If the buffer valve is used the loss curve of the pulsating current will rise like that of the alternating one. In the present case the pulsating current is similar to that represented in fig. 2 (*b*). In the previous case—that is, without the buffer valve—the line was terminated correctly only during half a cycle and by an open circuit during the other half cycle.

The curves of attenuation with respect of the frequency, represented in fig. 1, were obtained recording the attenuation at the far end of a loaded telephone cable, 68 km. long, of 13/10 mm. diameter, loaded with mHenry 177 coils

placed at one mile from each other. The attenuation is expressed in nepers.

The attenuation was greatly increased in the case of the pulsating current as the energy received at the far end of the cable was greatly reduced, considering that only one half-wave was sent into the cable—that is, only one-half of the energy generated by the oscillator which supplied the current sent into the cable. But considering the energy behind the rectifier, which really entered into the cable, the attenuation of the cable for low frequencies was about the same as for the alternating as well as for the pulsating current; for the higher frequencies the attenuation was much less in the case of the latter one.

Fig. 3.



I will now consider an alternating current the frequency of which was 982 cycles per second. This current entered the primary winding of a transformer with a transformation rate 1/1. The secondary winding of the transformer was inserted into a line. Another transformer 1/1 closed the line at its far end. An oscillograph inserted in the secondary winding of this transformer recorded the curve represented in fig. 4 (*a*) (Pl. IX.).

If behind the transformer at the near end of the line a rectifier—for instance, a valve rectifier—was put in series with one of the two line wires a pulsating current travelled on the line, as the rectifier allowed only one-half a wave to enter the line. I can consider this pulsating current decomposed into a direct component and into an alternating component of the same frequency as that of the

rectified alternating current. This alternating component, which entered the primary winding of the transformer placed at the far end of the line, gave rise to an alternating current in the secondary winding. In fact, the oscillograph inserted in the secondary winding of the transformer recorded the alternating current shown in fig. 4 (*b*) (Pl. IX.).

This current has the same frequency as that recorded formerly by the oscillograph when the full alternating current entered the line, only its amplitude was slightly reduced because of the resistance in the rectifier; but this amplitude was not a half of that recorded in the previous case, as it might seem at first, considering that only one-half the energy entered into the line.

Fig. 5 (*a*) (Pl. IX.) shows the oscillogram of 2650$\sim$ current transmitted through the line with its sinusoidal shape; and oscillogram of fig. 5 (*b*) (Pl. IX.) is that of the same rectified current recorded at the far end of the line.

For the transmission of the voice I can consider a complete sound decomposed by a Fourier series into some simple sounds represented by sinusoidal terms. Thus, by rectifying I will suppress one-half a wave of the simple alternating component currents. Fig. 6 (*a*) (Pl. X.) represents the vowel *i* pronounced in a steady manner before a quartz microphone by a young woman, and then transmitted through the line without changing the alternating current coming out from the microphone transformer, and recorded by the oscillograph placed at the far end of the line.

On the contrary, oscillogram of fig. 6 (*b*) (Pl. X.) represents the same vowel *i* transmitted through the line after the current has been transformed into a pulsating form, and then once more converted into an alternating one by means of the transformer at the far end of the line.

The peculiar shape of vowel *i* remained absolutely the same; there were still ten harmonics accompanying the fundamental frequency. We must consider that the man or woman who pronounces a vowel changes naturally from time to time the apparent shape and the loudness of his voice, and we must also remember that the oscillograms were taken at different times, because it was not possible to record them at the same instant.

Oscillogram of fig. 7 (*a*) (Pl. X.) represents the vowel *a*

pronounced by the same young woman and transmitted through the line with its full form; oscillogram of fig. 7 (*b*) (Pl. X.) shows the same *a* rectified at the near end of the line and then converted into alternating form at the far end of the line. Also by examining these oscillograms we can see how the peculiar shape of vowel *a* has not changed.

Beside the advantage of improving the quality of the voice transmitted through a heavily loaded cable, because it is possible to transmit a greater frequency range as the cut-off frequency of the cable will be a higher one, a commercial use of the pulsating current could be the transmission, through a loaded cable, of a pulsating current with a frequency higher than those strictly necessary for telephone communications, which are transmitted through a loaded cable at present time.

This method would allow the superposing on the telephone communication of a pulsating current with a teleprinter service. It would only be necessary to place at the far end of the cable a separating choke filter entering into the telephone apparatus, so as to prevent the higher frequency pulsating current, rendered alternating by means of the transformer at the far end of the cable, to be perceived by the telephone apparatus. It would also be necessary to put a high-pass filter entering the teleprinter circuit. If along the cable there were placed amplifier sections it would be sufficient to put behind the amplifier a high-pass filter so as to rectify the higher frequency current for the teleprinter service.

In such a case the currents for voice transmission could keep their alternating form, as it is usually done.

LVII. *The Transverse Thermomagnetic Effect: a Method of Measurement.* *By* G. Schmidt Nielsen, *cand. mag., Copenhagen* *.

THE Nernst-Ettingshausen effect † is partly caused by a direct action of the magnetic field on the electrons in the metal and partly by an indirect influence due to the polarized molecules.

* Communicated by Dr. N. Bohr.

† L. L. Campbell, 'Galvanomagnetic and Thermomagnetic Effects' (Longmans, Green & Co., 1923).

By investigating the ferromagnetic metals it is possible to distinguish between these two sources by using the following apparatus.

The essential part of the apparatus is a nickel tube Ni, 50 cm. long, with internal and external diameters equal to 1·8 cm. and 2·2 cm. respectively. A glass tube G, 1·0 cm. in diameter, is supported by two ebonite plugs P; thirty copper wires, connected in series, pass through this glass tube. These carry the magnetizing current, which thus passes thirty times along the axis of the tube. The space between the glass tube and the nickel tube contains water for cooling purposes. The water passes in and out through two brass tubes placed close to the ends of the nickel tube.

The nickel tube is surrounded by a jacket J, from which it is insulated by mica, the joint between them being made tight by a mixture of lead oxide and glycerine. The ends of the nickel tube are connected to a galvanometer by two nickel wires. Steam from a boiler is sent into J through two side-tubes, and the condensed water flows back to the boiler through a third tube.

When the apparatus has reached thermal equilibrium, the transfer of heat through the wall of the nickel tube is measured by measuring the rate of flow of the cooling water and its increase in temperature. The procedure in making the measurements was as follows:—A current (I amp.) was sent through the copper-wires, and the resulting P.D. between the ends of the nickel tube read on the galvanometer. I was increased from 0 to 15 amp., and then again decreased to 0 by steps of 1 amp.; after reversal of the current the procedure was repeated. The curve in fig. 1 represents the state of affairs after the process had been repeated several times; the abscissa represents the axial current in amps., the ordinate the transverse effect in absolute units. Curve 2 shows the magnetic induction $B=H+4\pi P$ as a function of the axial current. This was obtained in the usual way after dismounting the apparatus, eight windings of insulated wire being placed round the wall of the nickel tube in a longitudinal direction.

It is now assumed that the measured effect can be expressed by

$$E=Q'.P.l.\ \frac{\varDelta t}{\varDelta r}+Q.H.l.\ \frac{\varDelta t}{\varDelta r},$$

where E=P.D. between ends of nickel tube,

P=mean value of polarization in nickel,

H=mean value of magnetic force,

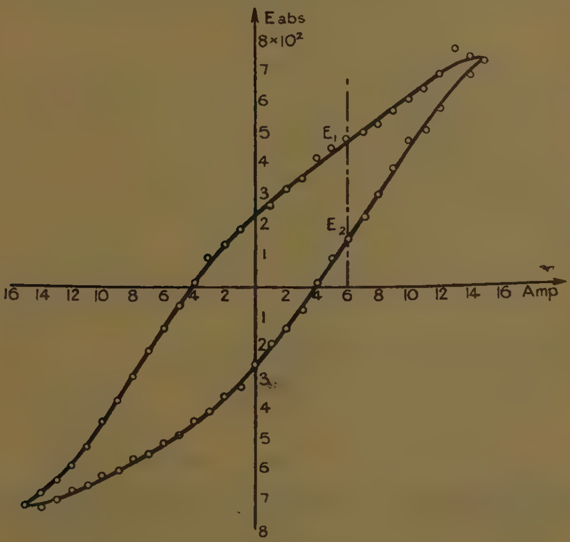
l=length of heated part of nickel tube,

$\frac{\Delta t}{\Delta r}$=temperature gradient in nickel,

Q′=coefficient for the effect due to polarization,

Q=coefficient for the effect due to magnetic field.

Fig. 1.



Where the curves 1 and 2 intersect the line I=const. we have

$$\frac{B_1-B_2}{4\pi}=P_1-P_2, \quad E_1-E_2=Q'(P_1-P_2)l\,\frac{\Delta t}{\Delta r},$$

or

$$Q'=\frac{E_1-E_2}{B_1-B_2}\cdot\frac{4\pi}{l}\cdot\frac{\Delta r}{\Delta t}.$$

Fig. 2.

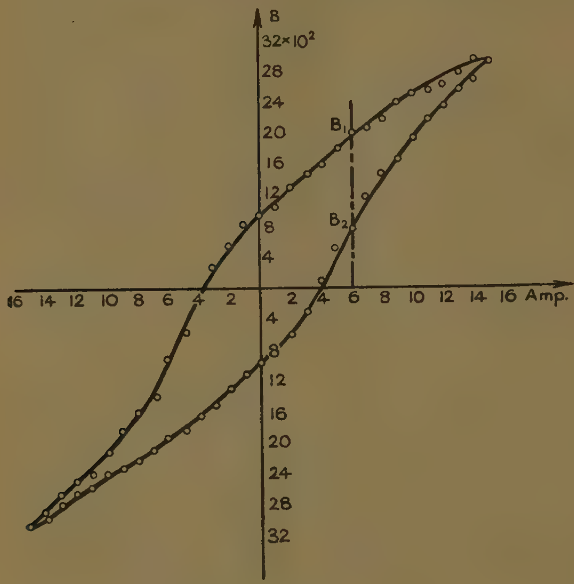
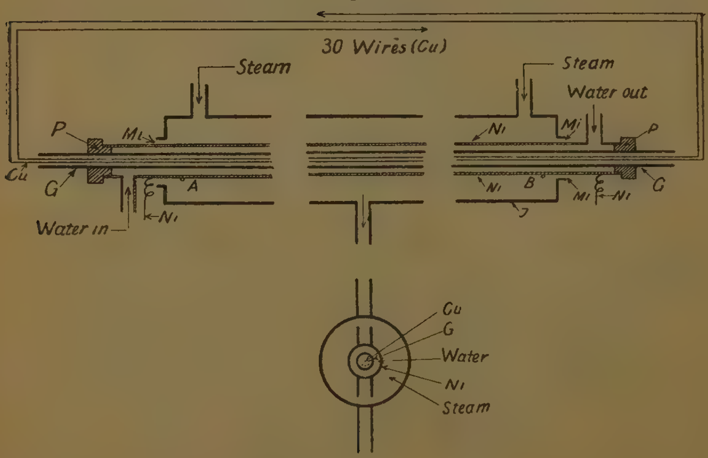

B
32×10²
28
24
20
16
12
8
4
B₁
B₂
16 14 12 10 8 6 4 2
2 4 6 8 10 12 14 16 Amp.
4
8
12
16
20
24
28
32

Fig. 3.


30 Wires (Cu)
Steam
Steam
Water out
P
Mi
Ni
Mi
P
Cu
G
A
Ni
B
Mi
Ni
G
J
Water in
Ni
Cu
G
Water
Ni
Steam

The transfer of heat through the wall of the nickel tube is

$$W=K.\frac{\Delta t}{\Delta r}.l.\sigma \text{ cal./sec.},$$

where

K=coefficient of conduction of heat,

σ=mean value of circumference of nickel tube.

From this we get

$$Q'=K.\frac{E_1-E_2}{B_1-B_2}.\frac{4\pi\sigma}{W}.$$

For the residual effect, when I=0 and H=0, we obtain

$$Q'=\frac{E.\Delta r}{P.l.\Delta t}=K.\frac{E.\sigma}{W.P}V.$$

The nickel tube used in the experiment gave $Q'=K.0{\cdot}15$, whereas the effect due to the direct action of the magnetic field in this case was too little for the determination of Q.

With this first apparatus, which was built for examination of the method, the P.D. was measured between the points shown in the figure. In order to get a better determination of l the wires must be soldered to the points A and B. Another drawback was that it could not support a sufficient current to obtain saturation of magnetization in the nickel tube.

This work is done at the Institute for Theoretical Physics in Copenhagen and will be continued occasionally. I wish to express my best thanks to Prof. Niels Bohr.

LVIII. *On the Influence of Molecular Structure on the Variation of Viscosity between the Melting- and Boiling-points.* By MARY D. WALLER, *B.Sc.*, *F.Inst.P.**

§ 1. INTRODUCTION.

THE purpose of the present study is to investigate further the relation which has been found to exist † between the symmetry of the molecule and the range of viscosity values in the liquid state. The term symmetry

* Communicated by A. C. Egerton.

† See preceding paper (Phil. Mag. xviii. pp. 505–16 (1934)).

includes not only the actual shape, but also the type of bond in the molecule.

In the first place the shape may be studied by comparing hydrocarbons consisting of open-chain and ring molecules respectively, where the atoms are of the same two kinds only, the polarities small, and the attractions between neighbouring molecules of the weak van der Waals type. In the previous paper it has been noticed that the viscosities of certain liquids consisting of spherical, or nearly spherical, or ring molecules do not vary greatly between the melting- and boiling-temperature, whereas the variation is much greater in the case of paraffins and ether, and this regardless of the various temperature intervals involved. This variation, as already mentioned, may be conveniently expressed by means of the characteristic ratio $r=\eta_f/\eta_b$ where η_f, η_b are the viscosities at the melting-point and boiling-point respectively. Other symmetrical molecules are considered below which confirm the conclusions already arrived at. Open-chain molecules, both of non-polar and polar types, and fused diatomic salts where the bond is ionic, are also considered in relation to the change of viscosity with temperature.

The connexion between symmetry and the range of viscosity in the liquid state is so marked that it is believed that the study of viscosities will prove to be a simple and powerful help in distinguishing between different geometrical isomers. A very marked periodic variation in r due to the number of carbon atoms in open chains has also been noticed, and also a difference in r between the even and odd carbon series.

The melting- and boiling-point viscosities η_f and η_b have in all cases been determined graphically by extrapolation from published data as described in the previous paper. Where the $(\log \eta,\ 1/\mathrm{T})$ relation (η=viscosity, T=absolute temperature) is linear, the extrapolation, even if extensive, is satisfactory. In some cases, however, only approximate values can be obtained. The variations in r which occur with changes of symmetry are fortunately so large that the approximate figures have sufficed for the purposes of the present paper. It is obviously most desirable that experimental values should be made of the viscosities at the melting- and boiling-temperatures, and that the values of r should be confirmed or amended as required. It is probable that this will lead to more

information regarding molecular structure. The viscosity values are usually given to two significant figures, and are expressed throughout in poises.

§ 2. Results.

The results are described below under headings relating to the various liquids. The sources from which some of the data have been derived have been stated in the previous paper; Bingham and Spooner's [1] recently collected fluidity data have made it possible to extend the data for the paraffin series by three members, and to include several more alcohols.

Table I.

Aliphatic Open-chain Hydrocarbons.

Substance.	t_f °C.	t_b °C.	t_b-t_f.	η_f.	η_b.	$\eta_f/\eta_b=r$.
Pentane	−131·5	36·2	167·7	·038	·0021	c. 18
Hexane........	− 94·3	69	163·3	·0194	·0021	c. 9
Heptane	− 90	98·4	188·4	·0279	·0021	13
Octane	− 56·5	124·6	181·1	·0189	·0021	9
Nonane........	− 51	150·6	201·6	·025	·0021	12
Decane	− 32	174	206	·0232	·0021	11
Undecane	− 26·5	197	223·5	·0295	·0020	16
Iso-pentane	−159·7	28	187·7	·122	·0021	c. 60
Ethers.						
Diethyl ether ..	−116·3	34·5	150·8	c. ·025	·0021	c. 12
Dipropyl ether	−122	89	211	c. ·1	·0021	c. 50

c. = very approximate.

Paraffins and Ethers.

The r values and relevant data for eight paraffins and for two ethers for which melting- and boiling-point, as well as viscosity, data are available are shown in Table I. The results have already been shown in graphical form in the preceding paper (*loc. cit.* fig. 4). The very rough nature of some of the figures is indicated in the table. The significant fact is that the values obtained for r are in all cases very much larger than for the metals and halogens examined and benzene.

Miscellaneous Low-ratio Liquids.

All the other liquids investigated (metals and halogens excluded, see preceding paper) which have yielded small values for r are included in Table II. No higher ratio appears in this list than that of thiophene, $r=4{\cdot}7$. It is significant that no long-chain molecules are included, and that the molecules are generally of high symmetry, when regard is paid to the length and type of bond and also the bond-angles. The

Table II.

Miscellaneous Organic Liquids with small Values of r.

Substance.	t_f °C.	t_b °C.	t_b-t_f.	η_f.	η_b.	$\eta_f/\eta_b=r$.
Benzene	5·5	79·6	74·1	·0082	·0034	2·2
Cyclo-pentane ..	−93·3	49·5	142·8	·0135	·0037	c. 3·7
Cyclo-hexane	6·5	81·4	74·9	·0124	·0039	3·2
Thiophene	−40	85	125	·016	·0034	4·7
Carbon tetra-chloride	−23	76·8	99·8	·020	·005	4·0
Chloroform	−63·5	61·2	124·7	·0175	·0039	c. 4·5
Bromoform	7·7	150·4	142·7	·023	·0057	4·1
Carbon dichloride..	−22·4	120·8	143·2	·0156	·0040	3·9
Methyl iodide	−66·1	42·6	108·7	·014	·0042	c. 3·34
Methyl sulphide ..	−83·2	36·2	119·4	·011	·0025	c. 4·4
Ethylene bromide..	10	137·1	121·7	·020	·0051	3·9
Ethylene chloride..	−35·3	83·7	119	·020	·0043	4·6
p-Xylene	13·2	137	123·8	·0074	·0024	3·1
Formic acid	8·4	100·5	92·7	·0234	·0054	4·25
Acetic acid	16·6	118·1	101·5	·0127	·0039	3·26

first three liquids in the table consist of flat rings and, containing as they do only carbon and hydrogen atoms, are particularly suitable for comparison with the open-chain paraffins. Thiophene, though containing sulphur, is a closed system, and includes two double-carbon bonds. The symmetry of the carbon tetrachloride molecule with four chlorine atoms situated at the corners of a tetrahedron is obvious. Carbon dichloride ($Cl_2-C=C-Cl_2$), $r=3{\cdot}9$, and methyl sulphide (CH_3-S-CH_3), $r=4{\cdot}4$, should be compared with water, $r=6{\cdot}4$ (see Table IV.). Water is, of course, known to be a complicated liquid,

Hund [2] has shown that the water molecule is triangular in structure on account of the strong polarizability of the O-atom in comparison with the H-atom. Ethylene bromide ($C_2H_4Br_2$), $Br-\overset{|}{\underset{|}{C}}=\overset{|}{\underset{|}{C}}-Br$, $r=3{\cdot}9$, may be compared with ethyl bromide (C_2H_5Br), $r=11{\cdot}6$ (Table IV.), as regards symmetry.

Formic acid (H—CO—OH) and acetic acid (CH_3—CO—OH) have also yielded low values for *r*; the higher members of the series behave like other long-chain molecules in having higher ratios; thus the values for propionic, butyric, and isobutyric acids are about 7·3, 8·1, and 16 respectively.

Benzene and Homologues.

Benzene and homologues (Table III.) are particularly instructive, for again only carbon and hydrogen atoms are

TABLE III.

Benzene and Homologues.

	Benzene, C_6H_6.	Toluene, C_7H_8.	Ethyl-benzene, C_8H_{10}.	*o*-Xylene, C_8H_{10}.	*m*-Xylene, C_8H_{10}.	*p*-Xylene, $C\ H_{10}$.
						
η_f	·0082	·055	·059	·019	·018	·00703
η_b	·0034	·0026	·0025	·0026	·0024	·0024
r	**2·2**	**21**	**24**	**6·6**	**7·5**	**3**
B	12·1	24·3	—	41·2	24·4	22·6
$\mu.10^{18}$ ..	0	0·4	—	0·6	0·4	0

r=ratio of melting-point to boiling-point viscosity. B=Kerr constant. μ=electric dipole moment.

involved. The rising values of *r*, a nearly eleven-fold increase, for the first three liquids, where the symmetry is diminishing, and the generally decreasing *r* values for the four isomeric modifications, C_8H_{10}, where the symmetry is increasing, is very striking. Values for the Kerr constant and for the electric dipole moments taken from

Wolf's chapter (p. 123) in Debye's 'Dipole Moment and Chemical Structure' are included for comparison. The manner in which these constants generally increase and decrease in the same sense as r does is very striking. The variations in r are, however, far more pronounced, and its great sensitiveness to alterations in molecular shape is made abundantly apparent. This suggests that its measurement would be a means of distinguishing between more or less symmetrical isomeric compounds which are hard to distinguish by other means. For example, the *cis-trans* isomerism due to the presence of a double bond can be distinguished with certainty (3) only by measurements of the dipole moments. The Author would hazard the opinion that the *cis*-forms would yield much higher values of r than the *trans*-forms.

Condensed Benzene Nuclei.

It is disappointing that data for naphthalene and anthracene, which have been so fully investigated in the solid state by X-rays, cannot be obtained. It would be particularly interesting to compare them with benzene. The few viscosity values available for naphthalene yield an approximate melting-point viscosity of ·0089 p., which is a little more than that of benzene, ·0082 p. As the boiling-point viscosity is likely to be nearly the same as that of benzene, it is probable that r has a rather higher value.

At the risk of digressing unduly, a curious relation which has been noticed between the normal melting- and boiling-points, t_f and t_b, of the three liquids has been tabulated below :—

 $t_f = 5{\cdot}5$ °C. $t_b = 79{\cdot}6$

 $t_f = 80{\cdot}1$ $t_b = 217{\cdot}9$

 $t_f = 218$ $t_b = 342$°C.

Thus at normal pressure benzene boils at the temperature that naphthalene melts, and naphthalene boils at the temperature that anthracene melts.

Substituted Hydrocarbons and Water.

An inspection of the data collected in Table IV, strengthens the conclusions already arrived at ; compare.

for example, the r values of methyl, ethyl, and propyl iodide respectivity. Points to be noticed are:—(1) Methyl

TABLE IV.

Substituted Hydrocarbons and Water.

Name.	η_f.	η_b.	r.
Propyl chloride	·043	·0028	15·3
Isopropyl chloride	·032	·0028	11·4
Isobutyl chloride	·12	·0029	41·4
Allyl chloride	·0172	·0027	6·37
Methylene chloride	·029	·0028	10·4
Ethylene chloride	·020	·0043	4·65
Ethylidene chloride	·039	·0035	11·2
Ethyl bromide	·043	·0037	11·6
Propyl bromide	·053	·0035	15·1
Isopropyl bromide	·027	·0034	9
Isobutyl bromide	·125	·0033	38
Allyl bromide............	·067	·0033	22·8
Ethylene bromide	·020	·0051	3·9
Propylene bromide	·076	·0049	15·5
Isobutylene bromide......	·21	·0048	43·7
Methyl iodide............	·014	·0042	3·34
Ethyl iodide	·049	·0039	12·5
Propyl iodide	·069	·0037	18·7
Isopropyl iodide	048	·0038	12·6
Isobutyl iodide	·091	·003	26
Allyl iodide	·064	·0036	17·8
Methyl sulphide..........	·011	·0025	4·4
Ethyl sulphide	·04	·0023	17·5
Carbon disulphide	·019	·0030	6·2
Fluoro-benzene	·0153	·0028	5·4
Chloro-benzene	·0277	·0028	8·1
Bromo-benzene	·033	·0033	10
Iodo-benzene	·043	·0035	12·3
Water	·018	·0028	6·4
Heavy water, $H_2{}^2O$	·020	c ·0028	c. 7

sulphide, r 4·4, $-\overset{|}{\underset{|}{C}}-S-\overset{|}{\underset{|}{C}}-$, as compared with carbon disulphide, $r=6{\cdot}2$, $S-C-S$. (2) The low ratios of the

methyl compounds, which were indeed included in the "low-ratio" Table II. (3) The ethylene chloride or symmetrical dichlorethane, $r=4{\cdot}65$, as compared with the ethylidene chloride or unsymmetrical dichlorethane, $r=11{\cdot}2$. (4) The fact that all the chain-molecules have high ratios and that the iso-compounds of a series do not necessarily have larger r values than the normal isomers on account of the shortening of the molecule: *cf.* Table IV. with pentane, $r=$c. 18; iso-pentane, $r=$c. 60 (see Table I.) (5) The r values for methylene chloride (CH_2Cl_2) and ethylene chloride ($ClCH_2CH_2Cl$), 10·4 and 4·65 respectively. If account be taken of the valency angle of methylene chloride, which according to Debye [5] is 150°, it becomes apparent that the law of symmetry is valid here as in all other cases investigated. (6) The effect of substitution of a halogen in the benzene ring ($r=2{\cdot}2$). We are no longer dealing with hydrocarbons, and the polarizations are, of course, greater.

Heavy water, $H^2{}_2O$, is included in the table for comparison with water. Lewis and Macdonald [6] give 3·8° C. and 101·42° C. as the melting- and boiling-points respectively, and also seven viscosity values between 5° C. and 35° C. The ($\log \eta$, 1/T) curves of both waters are somewhat concave to the 1/T axis, but converge towards the boiling-points when plotted in terms of corresponding temperatures. Accordingly the differences in viscosity are greater at the melting-points and least, or nil, at the boiling-points. The value of r is thus seen to be somewhat larger for heavy water than for ordinary water.

Alcohols.

The ($\log \eta$, 1/T) curves, though often concave to the 1/T axis, are generally capable of yielding rough values for r. Reference to Table V. shows that the variations are enormous as compared with any other liquids yet investigated. It is evident by comparison with the paraffins (Table I.) that it is the strongly polar nature of the open-chain molecule which is now of paramount importance, and that the temporary groupings between molecules postulated by various workers as occurring in the liquid state must be very pronounced at lower temperatures. It is therefore interesting to note that a qualitative agreement between the degree of X-ray

diffraction at room-temperature and the corresponding viscosities at 0° C. has been obtained by G. W. Stewart [7] in a study of twenty-two octyl alcohols. The correspondence might well be greater if the technical difficulties did not prevent measurements being made at corresponding temperatures.

It will be recalled that at room-temperature the lower members of the alcohols are mobile liquids, the middle ones more oily, while from dodecyl alcohol onwards they

TABLE V.

Alcohols.

Alcohols.	t_f °C.	t_b °C.	$t_b - t_f$.	η_f.	η_b.	r.
Methyl	− 97·8	64·5	162·3	c. 0·1	·0032	c. 30
Ethyl	−112	78	190	c. 1	·0048	c. 200
Propyl	−127	97·8	224·8	c. 30	·0049	c. 6000
Butyl	− 89·8	117·71	207·5	c. 3	·0037	c. 800
Amyl	− 78·5	137·9	216·4	—	—	—
Hexyl	− 51·6	155·8	221·4	—	—	—
Heptyl	− 34·6	175·8	238·4	c. 0·19	·0034	c. 56
Octyl	− 16·3	194	201·3	c. 0·4	·002	c. 200
Isopropyl ..	− 85·8	81·3	167	c. 3	·0053	c. 600
Isobutyl ..	−108	107·3	165	c. 43	·0047	c. 9000
Isoamyl	−117·2	130·5	155·8	c. 100	·0040	c. 24000
Tert. butyl ..	25·5	82·8	298·5	c. 0·04	·0058	c. 7·3
Tert. amyl ..	− 11·9	101·8	261·1	c. 0·2	·0043	c. 46·2
Allyl	−129	97	226	c. 6	·0039	c. 160·0

are solid. Although the variations of viscosity with temperature of most alcohols are so great, it still remains true that the boiling-point viscosities are, as with almost all other liquids, to be measured in hundredths of a poise. The values for the alcohols studied range only from ·002 p. to ·0058 p.

In view of the high r values of the normal, and the still higher values of the iso-alcohols, the much lower r values of the tertiary alcohols are very striking. Consideration of the constitutional formula, however, shows at once

how much less elongated and more symmetrical are the tertiary- than the iso-forms, so that even in the case of the strongly polar alcohols the influence of shape is profoundly evident. Isobutyl alcohol, $(CH_3)_2.CH.CH_2.OH$, $r=9000$, may be compared with tertiary-butyl alcohol $(CH_3)_3C.OH$, $r=7$; and similarly isoamyl alcohol, $(CH_3)_2.CH.CH_2.CH_2.OH$, $r=24000$, with tertiary-amyl alcohol, $(CH_3)_2.C(OH).CH_2.CH_3$, $r=46$. It is worth noting (see Table V., column 4) that, in spite of their high r values, the iso-alcohols remain liquid over a smaller range of temperature than do the tertiary alcohols. Whether or not it would be possible to detect anomalous viscosity effects at large velocities in some of the high viscosity alcohols, such as are found particularly in colloids consisting of elongated particles, is an interesting speculation.

Fused Salts.

The data relating to some twenty salts have been examined. In general the ($\log \eta$, $1/T$) graph is not a straight line, and the estimations of r obtained by extensive extrapolation are accordingly approximate.

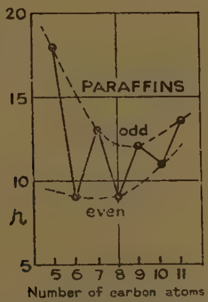
Lead bromide, silver chloride, and silver nitrate have ratios of about 10, 10, and 14 respectively, and the general results indicate that such ratios and higher ones are characteristic of diatomic ionic-bond molecules, in marked contrast to the low ratios of the homopolar diatomic halogen molecules. It must be emphasized that the available data in all cases extend over a restricted part of the liquid range only. The fact that the curves may be concave, as in the cases of lead chloride, sodium chloride, and sodium bromide; convex, as in the cases of potassium nitrate and potassium dichromate (which latter shows some irregularity); S-shaped, as in the case of lead bromide; and occasionally straight, as, for example, in the cases of silver chloride and sodium nitrate, indicate that complicated changes of viscosity are liable to occur as the temperature is altered. Even where straight lines are obtained these are over a fraction of the whole temperature range, and indicate only that for that range no transforming complications occur.

The irregular results obtained are in accordance with the fact that electric conductivity measurements are hard to interpret, and that surface-tension measurements indicate strong association [8].

§ 3. Hydrocarbon Odd and Even Chains.

When the r values of the paraffins are plotted against the number of carbon atoms in the molecule (see fig. 1), the fact that the paraffins containing an odd number of carbon atoms have higher r values than do the even series is very apparent. It is of course known that the odd and even series of hydrocarbon-chain compounds vary somewhat in other physical respects. Thus the paraffins show a smaller rise in their melting-points, when passing from an even to an odd, than from an odd to an even, numbered member of the series (see Table I.). This irregularity is most marked for the fatty acids, where a zig-zag figure is obtained, when the melting-points are

Fig. 1.



Variation in ratio of melting- to boiling-point viscosity r, and number of carbon atoms.

plotted against the number of carbon atoms, which is very similar in appearance to fig. 1.

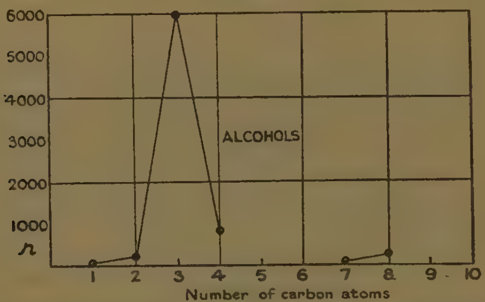
The higher melting-point viscosities of the odd series denotes a different grouping from those of the even series. Muller's [9] conclusions regarding hydrocarbon chains in the solid state are exceedingly interesting in this connexion. His X-ray analysis has shown that the chains consist of flat zig-zags which require a different packing at their ends according to whether they belong to the odd or the even series. In the even series the molecules are side-stepped at each junction, while the odd molecules remain along the same line. In the liquid state, therefore, we may suppose that temporary grouping is more pronounced in the odd than in the even series, and that this

explains the higher r values found for the odd as compared with the even series. The fact that, as Garner and Randall [10] have shown, the odd series of the fatty acids have lower heats of crystallization than the even series is also of interest in the present connexion, although their results refer to acids and not to paraffins.

§ 4. Flexibility of the Hydrocarbon Chain.

For the paraffins and the alcohols respectively, the number of liquids for which data are available has been sufficient to detect a periodic variation of r with the number of carbon atoms in the molecule, which seems, in the opinion of the Author, to provide evidence of the

Fig. 2.



Variation in ratio of melting- to boiling-point viscosity r, and number of carbon atoms.

flexibility of the hydrocarbon chain. Reference should be made to the two figures and to the relevant tables (I. and V.). In fig. 1 the r values of the normal paraffins are plotted against the number of carbon atoms. The highest r values are obtained for 5 and 11 (5+6) carbon atoms. The dotted curves drawn through the r values both of the even and of the odd series show a decrease, followed by an increase of r. It has been shown above that high r values belong to long and low r values to more nearly spherical molecules. Does this not therefore point to temporary configurations of the molecules which include bending of the chains into parts of helices ? The jump from $r=18$ to $r=9$, corresponding to 5 and 6 atoms respectively, is very marked.

In the case of the strongly polar alcohols (see fig. 2 and Table V.) there is a profound connexion between the number of carbon atoms in the molecule and the values of r. These increase two-hundredfold from the first to the third alcohol, diminish for the fourth, data are missing for the next two members, but with heptyl alcohol (6+1)C and octyl alcohol (6+2)C the figures are comparable with those obtained for methyl and ethyl alcohol.

A graphical examination of the melting- and boiling-points in relation to the number of carbon atoms, both of the paraffins and of the alcohols, does not show any evidence of this periodic change, except that the melting-points of the first three alcohols are in decreasing order, after which they continue to rise, without any obvious zig-zag alternation, between the odd and even series, as in the case of the acids.

It will be interesting to re-examine the above suggestion regarding evidence of the flexibility of the chain when more data are available. At any rate ample proof has been given of the extreme sensitiveness of this method of studying different molecules.

§ 5. Viscosities at the Boiling-point.

A few remarks only will be made on this subject in the present paper, and these will be restricted to hydrocarbons, and in relation only to the subject of molecular shape.

The fact that the paraffins all have the same viscosity at the boiling-point means that neither the molecular weight nor the length of the molecule affect its value as we pass from paraffin to paraffin. When the liquid is about to vaporize there is, presumably, no grouping, even of a temporary nature, between the molecules. The weak van der Waals forces which are responsible for the temporary groupings at lower temperatures are about to be overcome. We may well imagine that vaporization takes place by molecules which shoot out of the liquid like arrows. The cross-section only, and not the length, appears to be of account. The molecules of the liquid will glide past each other in directions parallel to their lengths. Similarly ring-molecules may be imagined as flat disks which glide parallel to their faces. With this picture in mind it is interesting to compare the boiling-point viscosities—paraffins, ·0021 ; benzene, ·0034 ;

cyclopentane, ·0037; cyclohexane, ·0039—with the cross-sections as determined by X-ray analysis, and by Adam [11] in his experiments on insoluble surface-films. The results of the former are conveniently summarized in Bragg's 'The Crystalline State'; Muller's researches on the hydrocarbon chains, p. 166, and Wierl's on cyclopentane, p. 265, show that the cross-sections of the molecules of the above organic liquids are in the same order as, and in rough quantitative agreement with, their boiling-point viscosities.

One other point may be noticed. Reference to Table III. shows that the boiling-point viscosities of the five benzene homologues, ·0026–·0024, are nearly equal to one another, but considerably less than that of benzene, ·0034. Whether or not this supports the view of those [12] who, in the study of the dipole moment of these liquids, postulate a straining of the benzene ring in the di-substituted benzene homologues it is, perhaps, too early to decide, but the idea is worth bearing in mind. The further study of boiling-point viscosities, and also their relation to gaseous boiling-point viscosities, should be of considerable interest. Extrapolation of Titani's [13] recent results for ten organic gases for which liquid viscosities were also available shows that the gaseous viscosities are from between 23 to 47 times less than the liquid viscosities, and also that the boiling-point viscosities of gaseous homologues are not equal to one another, as in the case of many liquid homologues; for example, the values for pentane and hexane have been found to be ·0000911 and ·0000762 respectively. Again, the value for ether, which in the liquid state coincides with the paraffins, is ·000078 in the gaseous state.

§ 6. General Conclusions.

1. The characteristic viscosity ratio r between the melting- and boiling-point viscosities is small (<4) for liquids whose molecules are either spherical or, at any rate, of high symmetry. This conclusion is based on a study of certain elements (metals and halogens), hydrocarbons (benzene, cyclopentane, cyclohexane), and various other liquids (carbon tetrachloride, carbon dichloride, etc.).

2. The characteristic viscosity ratio r is much higher for liquids whose molecules are elongated. The study of the paraffins, ethers, esters, aldehydes, ketones, and

alcohols, and various substituted hydrocarbons, supports this statement.

3. The value of r is very susceptible to change in molecular shape. This is well exemplified in the case of benzene and its homologues, in which r varies from 2·2 to 24, according to the degree of symmetry of the molecule. The variations in value according to symmetry are in general agreement with, but even more pronounced than, those of the dipole moment and Kerr constant. Other examples of the profound influence of symmetry on the value of r are given in the text.

4. The paraffins whose molecules contain an odd number of carbon atoms have higher r value than those which contain an even number of carbon atoms. This is in agreement with Muller's X-ray theory of the difference in the packing of odd and even hydrocarbon chains in the solid state. Even in the liquid state the temporary groupings which increase the viscosity will be different and more pronounced at lower temperatures in the odd than in the even series.

5. It is believed that evidence of the flexibility of the open hydrocarbon chain is provided by the suggestion of periodic variations in the value of r, as the number of carbon atoms increases by six in both the paraffin and alcohol series.

6. The above conclusions suggest a method of studying molecular structure. Whereas, on the one hand, X-ray analysis has been most extensively used for studying molecules in the solid state, and, on the other hand, the measurements of dipole moments and kinetic theory have been applied to gaining knowledge regarding molecules, more especially in the gaseous state, the present method promises to be of particular value in connexion with the liquid state. It should be able to confirm, and even to supplement the other methods of study. It will be of great interest to consider it in relation to conclusions based on the Raman effect and infra-red spectra.

7. The r values are so extremely susceptible to changes in molecular structure that, as suggested in the text, it should prove a powerful tool in the hands of the organic chemist in distinguishing between different stereo-isomeric compounds, which are otherwise hard to recognize.

8. It may be remarked that since the range of values of liquid viscosities is often much less at the boiling- than

at the melting-points, a knowledge of the viscosities at the latter temperature will sometimes be sufficient to give the same information regarding molecular structure as is given by the ratio *r*.

9. Finally, it may be emphasized that there are still many comparatively simple liquids for which viscosity measurements are entirely wanting, and that the number for which the measurements extend throughout the liquid range is very small. In many cases the melting-points, and in a fewer cases the boiling-points, are also not available.

References.

(1) 'Physics,' iv. p. 387 (1933).
(2) See Debye, 'The Dipole Moment and Chemical Structure,' pp. 30, 74 (1931).
(3) *Loc. cit.* p. 33.
(4) Bragg, 'The Crystalline State.—I.' p. 161 (1933).
(5) *Loc. cit.* (2) above, p. 67.
(6) J. Amer. Chem. Soc. lv. pp. 3057, 4730 (1933).
(7) Phys. Rev. xxxviii. p. 1575 (1931).
(8) Taylor, 'Physical Chemistry.—I.' p. 626 (1931).
(9) Proc. Roy. Soc., A, cxxiv. p. 317 (1929).
(10) J. Chem. Soc. cxxv. p. 883 (1924).
(11) 'Physics and Chemistry of Surfaces' (1929).
(12) *Loc. cit.* (2) above, p. 95.
(13) Chem. S. Japan Bull. viii. p. 255 (1933).

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LIX. *Ionization by Positive Ions in Helium.* *By* J. S. Townsend, *Wykeham Professor of Physics, Oxford, and* G. D. Yarnold, *Merton College, Oxford* *.

IT has been shown, in the treatise on 'Electricity in Gases,' that in order to obtain a consistent theory of ionization of gases in currents between parallel plate electrodes and in currents between coaxial cylinders it is necessary to assume that some of the molecules of the gas are ionized by single collisions of electrons and positive ions.

It is also well known that electrons are set free from the negative electrode by the impacts of positive ions. The

* Communicated by the Authors.

emission of electrons from the electrode due to this cause is not appreciable when the kinetic energy of the ions is small, but when the energy is above a certain value the emission increases with the velocity of impact of the ions.

The theory of the conductivity of gases, based on these hypotheses, is known as Townsend's theory, and was first deduced from experiments on diatomic gases.

Since then several investigations have been made at the Electrical Laboratory, Oxford, on the conductivity of monatomic gases, and it has been found that the processes of ionization are the same as in the diatomic gases, the coefficients of ionization being different for the different gases. Impurities may have large effects on the values of the coefficients of ionization, but the general form of the theory is the same for pure and impure gases.

There are also several recent theories of conductivity based on the hypothesis that there is no ionization by single collisions of electrons or positive ions with molecules of the gas under the conditions which are considered in these investigations. The supply of ions necessary to maintain a current is supposed to be derived from the action of metastable atoms which set free electrons from the electrodes or from impurities present in the gas. It is also supposed that positive ions do not ionize molecules of the gas but that electrons are set free from the negative electrode by the photoelectric effect of radiation emitted by the recombination of positive ions with electrons in the metal. It will be observed that under these hypotheses there would be no positive ions in pure monatomic gases, and in the positive discharges from a wire to a coaxial cylinder the only action of positive ions, if there were any in the gas, would be to set free electrons from the cylinder. This action would be independent of the radius of the cylinder and of the velocity of impact of the ions with the surface of the cylinder.

If the latter theories were accepted it would be impossible to obtain a reasonable explanation of the currents between parallel plates and the glow discharges from wires or points. It is not, however, our intention to discuss the inconsistencies of these theories, except to say that they are in complete disagreement with the results of experiments we have recently made on the ionization by positive ions in hydrogen and in helium.

The experiments with positive ions in helium which

are described in this paper are similar to those made with hydrogen which have already been published *.

The results obtained with both gases are in good agreement with the hypothesis adopted in the theory which is given in the treatise on ' Electricty in Gases.'

In this theory the molecules of the gas are ionized by single collisions of electrons and positive ions. The ionization by the electrons is represented by the coefficient α and that by the positive ions by the coefficient β. The coefficient γ is the ratio of the number of electrons set free from the negative electrode to the number of positive ions that impinge on it.

The coefficients α, β, and γ depend only on the electric force and the pressure of the gas, and are the same for small photoelectric currents as for currents of the order 10^{-5} ampere between parallel plates or for currents of 10^{-3} ampere in the positive columns of discharge-tubes.

The changes in the coefficients observed with larger currents are attributed to the disturbance of the electric field by charges in the gas or to the increase in temperature.

The fact that the same values of the coefficients α and β represent the processes of ionization very accurately in large and small currents shows that the particles that are ionized are normal atoms of the gas, since the number of abnormal atoms which may be generated by small photoelectric currents must be negligible.

The coefficient α may be determined from experiments on photoelectric currents between parallel plates when β and γ are small, but the experiments do not differentiate between the two effects of the position ions. In general † these experiments and the determinations of the sparking potentials between parallel plates give data for calculating the quantities $(\alpha-\beta)$ and $(\beta+\gamma\alpha)$, but the coefficients β and γ cannot be deduced separately.

Some indication of the relative values of β and γ is obtained from experiments on currents between coaxial cylinders which show that in positive discharges from an inner cylinder to an outer cylinder of large diameter the emission of electrons from the outer cylinder is negligible in comparison with the ionization of the gas by the collisions of positive ions.

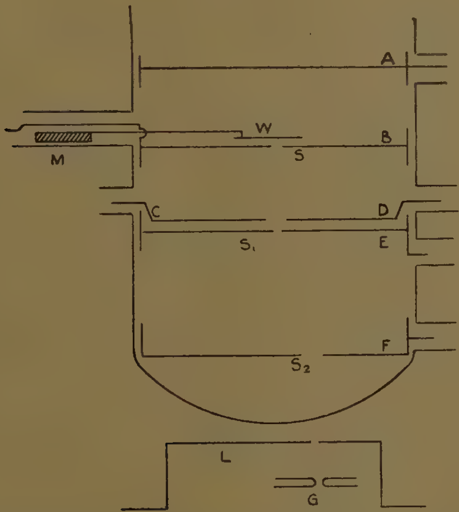
* J. S. Townsend and E. Llewellyn Jones, ' Nature,' 6th Sept. 1932, and Phil. Mag. ser. 7, xv. p. 282 (Feb. 1933).

† J. S. Townsend and S. P. McCallum, Phil. Mag. vi. p. 857 (Nov. 1928), and xvii. p. 678 (March 1934).

It was therefore concluded * that in currents between parallel plates at a distance apart (a) in a gas at pressure (p) "the most satisfactory hypothesis is that for the larger values of the product ap the principal effect of positive ions arises from ions generated in the gas. The relative importance of the ions set free from the negative electrode increases as the pressure diminishes."

If S be the distance between the plates when a spark discharge is obtained with a force X "it may be assumed

Fig. 1.



that the gas effect predominates when pS exceeds the value corresponding to the minimum sparking potential."

As it was impossible to deduce the actual values of β and γ from the earlier experiments a method was devised of finding the two coefficients separately.

The arrangement of the electrodes in the apparatus used for this purpose is shown in fig. 1. There were four circular electrodes, A, B, E, and F, 6 centimetres in diameter, and 2 semicircular electrodes, C and D, of smaller diameter. The distance between A and B was

* 'Electricity in Gases,' pp. 332 & 333.

2 centimetres, between B and E 2 centimetres, and between E and F 3 centimetres. The electrodes C and D were in the same plane, at a distance of 1 millimetre from E and with a gap between them 4 millimetres wide. The distance of the upper surfaces of C and D from the lower surface of B was 1·8 centimetres.

There was a slit S 4 millimetres wide in the centre of B, a slit S_1 2 millimetres wide in the centre of E, and a slit S_2 2 millimetres wide in the lower plate F at a distance of 6 millimetres from the centre. The slits were all parallel and each 1·5 centimetres long. The straight edges of the semicircular electrodes C and D were parallel to the slit in E and 2 millimetres from the central line. A shutter W operated by a magnet M was arranged to slide on the upper surface of B, so as to close the slit S when desired.

The four circular electrodes, A, B, E, and F, were fixed by means of rings at their edges to four vertical quartz rods (not shown in the figure), which served to hold them in their correct positions and to insulate them from one another. The semicircular electrodes C and D were held in position and insulated from the electrode E by thin plates of quartz. The whole apparatus was contained in a quartz cylinder, and the connexions to the electrodes were made through six side-tubes. Each side-tube was sealed by a graded joint to a short length of pyrex tubing through which passed a tungsten connecting wire.

The electrodes were of aluminium, as the insulation of the quartz supports is not affected when the aluminium is heated to a high temperature. The shutter W was of copper. In the apparatus used for the experiments on positive ions in hydrogen the electrodes and the shutter were of copper, but it was found that a conducting deposit was formed on the insulating supports of the electrodes when the apparatus was heated to a high temperature. As the emission coefficient γ was found for copper in the experiments with hydrogen, the same shutter was used in the experiments with helium, in order to compare the emission from copper due to the impacts of the positive ions of the two gases.

The positive ions which passed through the slits were generated in the gas between the electrodes E and F by the collisions of electrons with molecules of the gas.

The electrons were set free from the lower surface of

the electrode E by ultra-violet light from the spark-gap G in a condenser discharge. The light was limited to a narrow beam by a slit in the screen L, and entered the apparatus through the slit S_2 in the electrode F. The beam fell on a narrow area of the electrode E opposite the slit S_2, and none of the direct light passed through the slit S_1 to the electrode B.

The electrons set free by the light from the plate E move under an electric force to the plate F and ionize the gas. The stream of electrons spreads out, and some of the positive ions are generated opposite the slit S_1. After passing through the slit the positive ions move under an electric force X to the electrode B. The positive ion current passing through the slit S_1 may be adjusted to any desired value by changing the potential difference between the plates E and F. The positive current received by the electrode B was measured with an electrometer, and a current of sufficient intensity was obtained when the potential between the electrodes E and F was 30 or 40 volts less than the potential required to produce a spark discharge.

The action of the positive ions after passing through the slit S_1 was determined by measuring the negative currents received by the electrodes C and D, which were at the same potential as the electrode E, and comparing these currents with the positive current received by B.

When the force X between B and E was small no measurable current was received by C and D, although the electrode B received a large current as measured by the electrometer. When the force X was increased the negative currents received by C and D were measurable, and with large forces became nearly as great as the positive current received by B.

The electrons generated in the space between B and E diffused laterally, and a large proportion was received by the electrodes C and D, but the centre of the stream passed through the gap between them. In order to find the total current of electrons the stream was deflected by a magnetic force H so that the centre of the stream fell on the plate C. The proportion of the stream which passed through the gap between C and D was thus reduced, and was estimated by measuring the currents to the two plates C and D separately with different values of the

magnetic force. The mean deflexion of the stream was comparatively small, and with the magnetic force used in the experiments the number of electrons passing through the gap was about 10 per cent. of the number received by the electrodes. Thus the total number of electrons generated in the space between B and E was about 10 per cent. greater than the number received by the two plates C and D. The currents received by the two electrodes was therefore corrected by adding 10 per cent., in order to find the number of electrons generated in the space between B and E.

The current received by the electrodes C and D is due partly to the ionization produced by the positive ion stream in passing through the gas in the space between the electrodes B and E and partly to the electrons set free from the surface of the electrode B by the action of the positive ions.

It is possible to differentiate between the two actions of the positive ions by measuring the current to the electrodes C and D when the shutter W is drawn back and the slit S in the electrode B is uncovered. A very large proportion of the positive ions reaching the electrode B then passes through the slit S, and are collected by the electrode A, which is at a lower potential than the electrode B. The potential between A and B was adjusted so that the positive ions after passing through the slit S produced no appreciable ionization.

The positive current received by the electrode A was therefore the current passing through the slit S.

Let b be the positive current received by the electrode B when the slit S is closed, c the negative current received by the electrodes C and D (with 10 per cent. added to correct for the electrons passing through the gap between them). The positive ion current passing through the slit S_1 is therefore $(b-c)$.

Let a_1 be the positive current received by A, b_1 the positive current received by B, and c_1 the negative current received by C and D when the slit S is open. In this case the positive current through the slit S is a_1 and the positive current through the slit S_1 is $(a_1+b_1-c_1)$. Since the current through S_1 is not changed by opening the slit S, the quantities $(b-c)$ and $(b_1+a_1-c_1)$ are the same. This result was verified by experiment.

Every precaution was taken to eliminate impurities

from the helium used in these experiments. The methods adopted in the purification of the gas have been explained in previous papers *. The pure gas was admitted to the apparatus from a vessel containing charcoal cooled in liquid air, and the appartus was then heated to a high temperature by means of an electric furnace. The gas was pumped out, and the process was repeated several times with fresh gas until the spectrum of a high-frequency discharge in the upper part of the apparatus showed only lines due to pure helium. A trap containing metallic calcium was interposed between the apparatus and the greased tap through which gas was admitted in order to prevent the diffusion of impurities from the tap into the apparatus. It was then found that the gas in the apparatus remained perfectly pure for several days and that there was no change either in its spectrum or in its electrical conductivity.

The experiments were made with pressures of the gas from 1 to 4 millimetres.

With the pressure of 4 millimetres and a potential of 100 volts between B and E [$X=55$, $X/p=14$] the negative current c received by the plates C and D was not appreciable in comparison with the positive current b received by the plate B. With the same pressure and a potential of 112 volts between B and E the current c was about 2 per cent. of the current b [$X=62$, $X/p=15{\cdot}5$]. Since there is no appreciable negative charge received by C and D it must be concluded that the positive charge b is due to positive ions reaching the electrode B, and not to the emission of electrons, when the force X is small.

With larger values of the ratio X/p the current c increased in comparison with b and it was possible to measure the ratio of the currents more accurately. The results of the experiments with values of X/p from 17·5 to 66 are given in Table I. The currents c, b, c_1, b_1, and a_1 are in arbitrary units, in each case the positive ion current $(b-c)$ and $(a_1+b_1-c_1)$ through the slit S_1 being taken as 100.

The pressure p of the gas is given in millimetres and the potential V between the plates B and E in volts.

It will be observed that at each pressure the current c increases with the ratio X/p, and in the experiments at different pressures where this ratio is the same (*e.g.*, $p=2$,

* J. S. Townsend and S. P. McCallum, Phil. Mag. vi. p. 857 (Nov. 1928).

V=180, and p=1·5, V=135) the current c increases with the pressure. The increase in c obtained with an increase in X/p and p is due partly to the increase in the action of the positive ions and partly to the increase in the ionization by the electrons so generated.

It will also be noticed that a considerable charge c_1 is received by the electrodes C and D when the slit S is open, although the number of positive ions passing through the slit S is approximately the same as the number passing through the slit S_1.

With values of X/p less than 50 the coefficient of emission γ from the electrode B is less than one-twentieth, so that

Table I.

p.	V.	X/p.	c.	b.	c_1.	b_1.	a_1.
4	128	17·6	5	105	—	—	—
4	160	22	21	121	—	—	—
4	192	26·4	54	154	24	22	102
3	144	26·4	18	118	—	—	—
3	180	33	72	172	28	25	103
2	120	33	19	119	—	—	—
2	160	44	65	165	30	17	113
2	180	50	118	218	44	34	110
1·5	135	50	35	135	13	12	101
1·5	180	66	123	223	41	40	101
1	120	66	24	124	6	12	94

the positive currents b and b_1 are very nearly the same as the numbers of positive ions reaching the electrode. If the current c received by the electrodes C and D were entirely due to the emission from B the ratio of the currents c_1/c would be the same as the ratio of the currents b_1/b. The numbers given in the table show that by opening the slit S the current received by the electrode B is reduced in a much greater proportion than the current received by the electrodes C and D. For instance, in the experiments at a pressure of 3 millimetres and X/p=33 the ratio of the currents c_1/c is about 0·4 and the ratio of the currents b_1/b about 0·14.

Except at the lowest pressure the current c_1 is greater than b_1. This shows decisively that the current c cannot be altogether due to emission from the electrode B.

Under these conditions it is necessary to attribute a large part of the current c received by the electrodes C and D when the slit S is closed to the ionization of the gas by the primary stream of positive ions through the slit S_1.

In order to estimate what proportion of the current c may be attributed to the different actions of the positive ions the same notation may be used as in the previous paper. Let n be the number of positive ions passing through the slit S_1 and βn the number of molecules ionized by the positive ions in moving 1 centimetre through the gas. Since the upper surfaces of the electrodes C and D are 1·8 centimetres from the lower surface of B, the number of electrons generated in the gas by the primary stream of positive ions is $1{\cdot}8 \times \beta n$. In moving towards the electrodes C and D these electrons ionize the gas, so that the current c is amplified. If the amplifying factor is x, the total charge conveyed to the electrodes C and D by this process is $1{\cdot}8\, x\beta n$.

Similarly, let γn be the number of electrons set free from the electrode B by the impacts of the primary positive ion stream and $y\gamma n$ the number which arrive at the electrodes C and D. The amplifying factor y is greater than x because the electrons set free from B traverse the whole distance between B and the electrodes C and D, whereas the majority of those generated in the gas by the positive ions traverse shorter distances.

The current c is therefore made up of two parts, one proportional to β and the other proportional to γ, and may be written as

$$c = 1{\cdot}8 \times x\beta(b-c) + y\gamma(b-c), \quad . \quad . \quad . \quad . \quad (1)$$

since the number n is proportional to $(b-c)$.

When the slit S is opened there is no change in the action of the positive ions in passing through the gas, but a number of positive ions proportional to a_1 passes through the slit. The corresponding reduction in the current received by the electrodes C and D is $c-c_1$, and is given by the equation

$$c - c_1 = y\gamma a_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The quantity $y\gamma$ is thus found directly from the experiments, and the following equation for $1{\cdot}8 \times x\beta$ is obtained from (1):

$$1{\cdot}8 \times x\beta = c/(b-c) - (c-c_1)/a_1. \quad . \quad . \quad . \quad . \quad (3)$$

The values obtained for $1{\cdot}8\,x\beta$ and $y\gamma$ in the different experiments are given in Table II., omitting those which are too small to be measured accurately.

The amplifying factors are not quite the same as they would be if the electrodes were close together and the field were not disturbed by the flanges at the edges. The factor y is more easily determined than x, since it represents the ratio of the number of electrons reaching the plates C and D to the number set free from B. The factor y may be found by measuring the currents received by the plates C and D when ultra-violet light is allowed to fall on the plate B. Another set of experiments were therefore

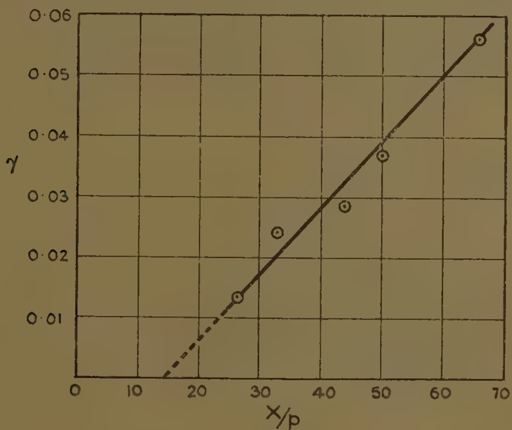
TABLE II.

p.	V.	X/p.	$1{\cdot}8x\beta + y\gamma$.	$1{\cdot}8x\beta$.	$y\gamma$.	y.	γ.
4	128	17·6	0·05	—	—	—	—
4	160	22	0·21	—	—	8·7	—
4	192	26·4	0·54	0·25	0·29	22	0·013
3	144	26·4	0·18	—	—	6·6	—
3	180	33	0·72	0·29	0·43	17·7	0·024
2	120	33	0·19	—	—	4·3	—
2	160	44	0·65	0·34	0·31	10·8	0·029
2	180	50	1·18	0·51	0·67	18·7	0·036
1·5	135	50	0·35	—	0·22	5·6	0·039
1·5	180	66	1·23	0·42	0·81	14·3	0·057
1	120	66	0·24	—	0·19	3·5	0·055

made by displacing the spark-gap G to the side, so that a beam of light from it passed through the slits S_2 and S_1 and fell on the lower surface of B. Under these conditions the current received by C and D is yN, where N is the number of electrons set free from the electrode B by the light and which move away from the electrode. When the electric force is small the number of electrons that are set free by the ultra-violet light and move away from the plate B increases as the force X increases, but it may be assumed that all the electrons which are set free by the light move from B to the opposite electrodes when the potential difference between the plates B and E is 80 volts or more. The number of electrons $y_1 \times$N arriving at the electrodes C and D when the potential between

B and E is 80 volts is greater than the number N which leave the plate B, and the amplifying factor y_1 may be found with sufficient accuracy from the results of previous experiments with parallel plates *. By measuring the current received by the electrodes C and D when the potential is 80 volts it is possible to determine the number of electrons which are liberated by the ultra-violet light falling on the electrode B. The amplifying factors y, corresponding to potential differences between the electrodes B and E greater than 80 volts, may then be found by measuring the currents received by the electrodes C and D when the ultra-violet light is allowed to fall on B.

Fig. 2.



The amplifying factors y determined in this way for the different forces and pressures are given in the seventh column of Table II., and the values deduced for the coefficient of emission γ in the different experiments are given in the last column.

It will be seen that the coefficient γ increases as the ratio X/p increases. In the two cases where it was possible to determine the coefficient γ at different pressures and with the same value of the ratio X/p the agreement is good. Thus, with $X/p=50$ the values 0·036 and 0·039 were found for γ, and with $X=66$ the values 0·057 and

* J. S. Townsend and S. P. McCallum, Phil. Mag. xvii. p. 678 (March 1934).

0·055 were found. The coefficient of emission γ is shown as a function of the ratio X/p in fig. 2.

The points are approximately on a straight line which cuts the axis at the point $X/p=14$, at which it was found that only a very small current was received by the electrodes C and D although the electrode B received a large current of positive ions.

The values of β cannot be determined as accurately as the values of γ from these experiments, as it is difficult to estimate the amplifying factors x. In the previous investigations with parallel plates of nickel it was shown that the currents may be expressed in terms of $(\alpha-\beta)$ and $(\beta+\gamma\alpha)$. Assuming the values of γ to be of the same order for the electrodes used in these experiments, the amplifying factors x may be calculated approximately and the values of β determined from the values of $1{\cdot}8\times x\beta$ given in Table II. The values of β/p thus obtained for

Table III.

X/p	26·4	33	44	50	66
β/p	0·01	0·014	0·03	0·033	0·04
$(\beta+\gamma\alpha)/p$......	0·014	0·024	0·046	0·057	0·08

the five different values of X/p are given in Table III. The values of $(\beta+\gamma\alpha)/p$ obtained from these experiments, where γ denotes the emission from copper, are given in the third line of Table III. They may be compared with

Table IV.

X/p	22·2	25	32·5	59·3
$(\beta+\gamma\alpha)/p$......	0·0145	0·0185	0·031	0·074

the values of $(\beta+\gamma\alpha)/p$ given in Table IV. which were obtained in the previous experiments * on the photoelectric currents between parallel nickel plates. The results given in Table III. and the curve in fig. 2 show that the coefficient of ionization β increases in comparison with γ as the pressure is increased.

It must be concluded that the general hypotheses on which the original theory of discharges was based apply to helium as well as to diatomic gases.

* J. S. Townsend and S. P. McCallum, *loc. cit.*

LX. *The Mercury Arc at Liquid Air Temperature.*

By F. H. Newman, *D.Sc., F.Inst.P., Professor of Physics, University College, Exeter* *.

THE mechanism of the mercury arc in which the cathode is at a relatively low temperature may be explained by the Langmuir theory of a very high electric field in the cathode neighbourhood, the same theory applying also to all arcs in which the cathode is not hot enough for appreciable thermionic emission; whereas a high temperature cathode arc will depend for its action, in part at least, upon thermionic emission. Whichever theory holds, between 90 and 100 per cent. of the current at the cathode should be carried by electrons; any smaller fraction would require a positive ion current, which is not likely to be found owing to the inefficiency of ionization by impacting electrons whose energy approximates to the ionization potential, viz., 10·4 volts for mercury vapour. In addition the fraction of the current carried by electrons probably depends upon the vapour pressure. Experiments with the mercury arc at liquid air temperature support this idea.

The arc-discharge tube in the present work was constructed of pyrex glass with sealed-in tungsten electrodes. The cathode was a pool of mercury, the anode a tungsten rod, and a third tungsten electrode was used to pass a momentary electrical discharge from a coil by means of which the arc was started. The distance between the mercury surface and the lower extremity of the anode was 1·5 cm., and the entire tube, which could be immersed in a liquid air container, was evacuated to about 10^{-2} mm. of mercury, since a small gas pressure is required for the electrical discharge; the gas present was hydrogen. A potential difference of 100 volts D.C. was applied between the anode and cathode—mercury the cathode,—a variable resistance being included in the circuit to regulate the current (5 amperes). Passing a momentary electrical discharge between the mercury and the third electrode, with this potential difference applied across the cathode and anode, started the arc, and the latter was maintained indefinitely even when surrounded with liquid air—the mercury being solidified.

* Communicated by the Author.

The most interesting feature of the arc was the almost complete absence of luminosity, not only between the mercury and anode but throughout the length of the tube immersed in liquid air, although the characteristic mercury radiation appeared in that part of the tube not immersed and extended along a portion of the connecting tubing. Spectroscopic investigation of this radiation showed the ordinary mercury lines together with a faint continuous band spectrum in the red. As the arc tube was lowered into the liquid air the luminous radiation retreated along the tube away from the Dewar vessel, and as the tube was raised from the container the radiation returned towards the solid mercury.

On first starting the arc there was luminosity between the electrodes characteristic of hydrogen—the residual gas—but after a few seconds this disappeared, and the mercury radiation (fairly intense) appeared in the upper part of the tube. The end of the tungsten anode quickly became almost incandescent, but after the arc had been burning for a short interval the anode temperature decreased, although still remaining higher than that corresponding to the arc at ordinary temperatures.

The small brilliant cathode spot observed in the mercury arc at ordinary temperatures also existed at the liquid air temperature, but instead of moving over all parts of the cathode surface in an erratic manner it revolved at high speed around the solid mercury-glass junction. This revolution is attributed to the action of the magnetic field produced by the electron current from the cathode. Owing to the uneven nature of the solid mercury surface the cathode spot would be expected to wander over the surface in an erratic manner, but such was not the case.

The increased anode temperature when liquid air surrounds the arc tube is attributed to the bombardment of the anode with electrons from the cathode. The vapour pressure between the electrodes is exceedingly low and the electrons lose little energy by molecular impacts during their travel from cathode to anode. It should be mentioned here that after the arc had been started the residual gas pressure was reduced, by pumping, to 10^{-4} mm. of mercury. Mercury vapour is absent throughout that part of the tube immersed in the liquid air, although present in the upper portion, where the temperature is certainly higher. The very faint luminosity

between anode and cathode is due to the fact that practically all of the current is carried by the electrons from the cathode and comparatively few positive ions are produced in this space. These electrons thus traverse the tube with high speed, and, bombarding the mercury atoms in the upper part of the tube, produce the characteristic mercury radiation. The fact that this radiation moves along the tube as the latter is raised or lowered in the liquid air confirms this theory.

The spectrum of the cathode bright spot could not be photographed, but examined visually it consisted of the usual mercury lines, together with a fairly bright continuous band throughout the whole of the visible spectrum.

Unfortunately these experiments do not give rise to any conclusive evidence for or against the rival "thermionic emission" and "high field" theories of the arc. The mercury remains solid, but there is the cathode bright spot, which may be at a considerable temperature. A non-luminous stream of mercury vapour appears to be emitted from this spot, otherwise one cannot account for the considerable amount of mercury vapour which must exist in the upper part of the tube. The liquid air will reduce the vapour pressure to a very small value unless the vapour is being continually replenished from the cathode spot; although the tube is immersed in liquid air, since the anode remains hot the temperature in the space between anode and cathode must be considerably higher than that corresponding to liquid air. Thus the stream of mercury vapour, emitted from the cathode with a high velocity, passes to the upper part of the tube, where in the course of time a considerable amount of condensed mercury collects.

Apart from the interesting fact that a mercury arc can be started and maintained at liquid air temperature, these experiments show that the current is carried almost entirely by electrons and that the cathode bright spot exists on a solid cathode even when the temperature of the electrode is low. It cannot be assumed that the temperature of the solid mercury was that corresponding to liquid air, but it must have been well below the solidification point, since on removal of the liquid air, the arc still passing, a considerable time interval elapsed before the mercury liquified. Thus in explaining the

mechanism of the arc it is not necessary to assume that the whole of the cathode is at a high temperature, and in the present work, since the mercury surface remained solid, it is probable that the temperature of the cathode spot is well below that obtained or assumed by many workers.

LXI. *The Polar Form of the ker and kei Functions, with Applications to Eddy Current Heating.* *By* N. W. McLachlan, *D.Sc., M.I.E.E., and* A. L. Meyers, *B.Sc., A.M.I.E.E.**

1. *Introduction.*

In solving problems relating to eddy current losses in straight cylindrical wires, cylindrical tubes, radio frequency coils, eddy current furnaces, and the like it is convenient to introduce Kelvin's ber and bei functions †. These are generally used in cartesian form, but in many cases it may be preferable to employ the polar form.

From the definitions

$$J_0(z\angle 45^\circ)=J_0(zi^{-1/2})=\mathrm{ber}\,z+i\,\mathrm{bei}\,z\,; \quad . \quad . \quad . \quad . \quad (1)$$

$$J_1(z\angle 45^\circ)=J_1(zi^{-1/2})=-(\mathrm{ber}_1\,z+i\,\mathrm{bei}_1\,z)\,; \quad . \quad . \quad (2)$$

$$\rho_0=\sqrt{(\mathrm{ber}^2\,z+\mathrm{bei}^2\,z)}\,;\;\; \theta_0=\tan^{-1}\frac{\mathrm{bei}\,z}{\mathrm{ber}\,z}; \quad . \quad . \quad (3)$$

$$\rho_1=\sqrt{(\mathrm{ber}_1{}^2 z+\mathrm{bei}_1{}^2 z)}\,;\;\; \theta_1=-\pi+\tan^{-1}\frac{\mathrm{bei}_1\,z}{\mathrm{ber}_1\,z}; \quad (4)$$

A. E. Kennelly, F. A. Laws, and P. H. Pierce ‡ tabulated ρ_0, θ_0, ρ_1, and θ_1 from $z=0$ to 10 with interval 0·1. These tables are reprinted in the British Association Report, p. 293 (1923), whilst similar tables are given in the 1933 edition of Jahnke and Emde's 'Funktionentafeln,' as also is the polar form of $H_0^{(1)}(zi^{1/2})=-\frac{2}{\pi}iK_0(zi^{-1/2})$ and

* Communicated by the Authors.

† 'Mathematical and Physical Papers,' iii. p. 491. The functions had previously been used with a different notation by Heaviside, 'Electrician,' xii. p. 585 (1884); 'Electrical Papers,' i. p. 360.

‡ Trans. Am.I.E.E. xxxiv. p. 1953 (1915). The notation used herein is that of McLachlan's 'Bessel Functions for Engineers' (Clarendon Press, 1934), so M replaces ρ.

its first derivative. In terms of the ker and kei functions

$$H_0^{(1)}(zi^{1/2})=\frac{2}{\pi}\sqrt{(\mathrm{ker}^2 z+\mathrm{kei}^2 z)}\;\underline{\big/-\tfrac{1}{2}\pi-\tan^{-1}\frac{\mathrm{kei}\, z}{\mathrm{ker}\, z}},$$

whilst

$$H_1^{(1)}(zi^{1/2})=\frac{2}{\pi}\sqrt{(\mathrm{ker}'^2 z+\mathrm{kei}'^2 z)}\;\underline{\big/\tfrac{1}{4}\pi-\tan^{-1}\frac{\mathrm{kei}'\, z}{\mathrm{ker}'\, z}},$$

which differ from the definitions (7) to (10) given below.

The original definition of the ber and bei functions is $J_0(z\epsilon)=\mathrm{ber}\, z+i\,\mathrm{ber}\, z$, where $\epsilon^2=-i$, the positive root being chosen * so that $\epsilon=+i\sqrt{i}=i^{3/2}$. This convention is used by a number of writers, *e. g.*, H. B. Dwight, A. Russell, H. G. Savidge, G. N. Watson, C. S. Whitehead †, and is the one we shall follow herein. Thus we have

$$J_\nu(zi^{3/2})=\mathrm{ber}_\nu\, z+i\,\mathrm{bei}_\nu\, z=M_\nu(z)e^{i\theta_\nu(z)}. \quad . \quad . \quad (5)$$

If ν is an even integer, $J_\nu(y)=J_\nu(-y)$, and since $i^{3/2}=-i^{-1/2}$, it follows that

$$J_0(zi^{3/2})=J_0(zi^{-1/2}). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

2. *Polar Formulæ for the ker and kei Functions* ‡.

We define the ker and kei functions by the relationships

$$i^{-\nu}K_\nu(zi^{1/2})=\mathrm{ker}_\nu\, z+i\,\mathrm{kei}_\nu\, z=N_\nu(z)e^{i\phi_\nu(z)}, \quad . \quad (7)$$

or

$$K_\nu(zi^{1/2})=N_\nu(z)e^{i\{\phi_\nu(z)+\frac{1}{2}\nu\pi\}}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$N_\nu(z)=\sqrt{(\mathrm{ker}_\nu{}^2 z+\mathrm{kei}_\nu{}^2 z)}, \quad . \quad . \quad . \quad . \quad (9)$$

and

$$\phi_\nu(z)=\tan^{-1}\frac{\mathrm{kei}_\nu\, z}{\mathrm{ker}_\nu\, z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

In particular

$$K_0(zi^{1/2})=\mathrm{ker}\, z+i\,\mathrm{kei}\, z, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$N_0(z)=\sqrt{(\mathrm{ker}^2 z+\mathrm{kei}^2 z)}, \quad . \quad . \quad . \quad (12)$$

$$\phi_0(z)=\tan^{-1}\frac{\mathrm{kei}\, z}{\mathrm{ker}\, z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$K_1(zi^{1/2})=N_1(z)e^{i\{\phi_1(z)+\frac{1}{2}\pi\}}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

* Kennelly, Laws, and Pierce chose the negative root, viz., $-i\sqrt{i}=i^{-1/2}$.

† Quart. Journ. Math. xlii. p. 316 (1911).

‡ 'Bessel Functions for Engineers,' Chap. 8.

From (11) $\frac{d}{dz}\{K_0(zi^{1/2})\}=\text{ker}'z+i\,\text{kei}'z,$

so $$i^{1/2}K_0'(zi^{1/2})=\text{ker}'z+i\,\text{kei}'z,$$

and $$-i^{1/2}K_1(zi^{1/2})=i^{-3/2}K_1(zi^{1/2})=\text{ker}'z+i\,\text{kei}'z. \quad . \quad (15)$$

Substituting from (14) in (15) we obtain

$$N_1(z)e^{i\{\phi_1(z)-\frac{1}{4}\pi\}}=\text{ker}'z+i\,\text{kei}'z,$$

so $$N_1(z)=\sqrt{(\text{ker}'^2z+\text{kei}'^2z)}, \quad . \quad (16)$$

and $$\tan\{\phi_1(z)-\tfrac{1}{4}\pi\}=\frac{\text{kei}'z}{\text{ker}'z}, \quad . \quad . \quad . \quad . \quad . \quad (17)$$

or $$\phi_1(z)=\tan^{-1}\frac{\text{kei}'z}{\text{ker}'z}+\tfrac{1}{4}\pi. \quad . \quad . \quad (18)$$

Using the asymptotic expansions, it can be shown that when $z>8$,

$$N_0(z)\doteqdot\sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{1-\frac{1}{8\sqrt{2z}}+\frac{1}{256z^2}+\frac{133}{2048\sqrt{2z^3}}\right\}, \quad . \quad . \quad . \quad (19)$$

$$\phi_0(z)\doteqdot-\frac{z}{\sqrt{2}}-\frac{1}{8}\pi+\frac{1}{8\sqrt{2z}}-\frac{1}{16z^2}+\frac{25}{384\sqrt{2z^3}}\text{ radians}, \quad . \quad . \quad . \quad (20)$$

$$\doteqdot-22\cdot5-40\cdot514z+\frac{5\cdot064}{z}-\frac{3\cdot581}{z^2}+\frac{2\cdot637}{z^3}\text{ degrees}, \quad . \quad . \quad . \quad (21)$$

$$N_1(z)\doteqdot\sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\left\{1+\frac{3}{8\sqrt{2z}}+\frac{9}{256z^2}-\frac{327}{2048\sqrt{2z^3}}\right\}, \quad . \quad . \quad . \quad (22)$$

$$\phi_1(z)\doteqdot-\frac{z}{\sqrt{2}}-\frac{5}{8}\pi-\frac{3}{8\sqrt{2z}}+\frac{3}{16z^2}-\frac{21}{128\sqrt{2z^3}}\text{ radians}, \quad . \quad . \quad . \quad . \quad (23)$$

$$\doteqdot 247\cdot5-40\cdot514z-\frac{15\cdot193}{z}+\frac{10\cdot743}{z^2}-\frac{6\cdot646}{z^3}\text{ degrees}. \quad . \quad . \quad . \quad (24)$$

When $0\leq z<1\cdot0$

$$\text{ker}\,z\doteqdot A_0+\frac{1}{4}\pi\left(\frac{1}{2}z\right)^2-\frac{1}{4}\left(A_0+\frac{3}{2}\right)\left(\frac{1}{2}z\right)^4-\frac{\pi}{144}\left(\frac{1}{2}z\right)^6$$
$$+\frac{1}{576}\left(A_0+\frac{25}{12}\right)\left(\frac{1}{2}z\right)^8, \quad (25)$$

$$\text{kei}\, z \doteqdot -\frac{1}{4}\pi+(A_0+1)\left(\frac{1}{2}z\right)^2+\frac{\pi}{16}\left(\frac{1}{2}z\right)^4 -\frac{1}{36}\left(A_0+\frac{11}{6}\right)\left(\frac{1}{2}z\right)^6-\frac{\pi}{2304}\left(\frac{1}{2}z\right)^8, \quad (26)$$

$$\text{ker}'z \doteqdot -\frac{1}{z}+\frac{1}{4}\pi\left(\frac{1}{2}z\right)-\frac{1}{2}\left(A_0+\frac{5}{4}\right)\left(\frac{1}{2}z\right)^3-\frac{\pi}{48}\left(\frac{1}{2}z\right)^5 +\frac{1}{144}\left(A_0+\frac{47}{24}\right)\left(\frac{1}{2}z\right)^7, \quad (27)$$

$$\text{kei}'z \doteqdot \left(A_0+\frac{1}{2}\right)\left(\frac{1}{2}z\right)+\frac{1}{8}\pi\left(\frac{1}{2}z\right)^3-\frac{1}{12}\left(A_0+\frac{5}{3}\right)\left(\frac{1}{2}z\right)^5 -\frac{\pi}{576}\left(\frac{1}{2}z\right)^7, \quad . \quad . \quad . \quad (28)$$

where $A_0=(0\cdot1159315\ldots\ldots-\log_e z)$.

When $0\leq z<0\cdot2$

$$N_0(z)\doteqdot B_0-\frac{\pi}{4B_0}\left(\frac{1}{2}z\right)^2 +\frac{\left\{A_0^2+\frac{5}{2}A_0+2+\frac{\pi^2}{16}\left(1-\frac{2}{B_0^2}\right)\right\}\left(\frac{1}{2}z\right)^4}{4B_0}, \quad (29)$$

where $B_0^2=A_0^2+\dfrac{\pi^2}{16}$.

$$\phi_0(z)\doteqdot-\tan^{-1}\frac{\pi}{4A_0}+\left(1+\frac{A_0}{B_0^2}\right)\left(\frac{1}{2}z\right)^2 +\frac{\pi}{4B_0^2}\left(\frac{5}{8}+\frac{A_0}{B_0^2}\right)\left(\frac{1}{2}z\right)^4 \text{ radians}, \quad (30)$$

$$N_1(z)\doteqdot\frac{1}{z}-\frac{\pi}{4}\left(\frac{1}{2}z\right)+\left(A_0^2+\frac{3}{2}A_0+\frac{7}{8}\right)\left(\frac{1}{2}z\right)^3, \quad . \quad . \quad (31)$$

$$\phi_1(z)\doteqdot\frac{5}{4}\pi-2\left(A_0+\frac{1}{2}\right)\left(\frac{1}{2}z\right)^2-\pi\left(A_0+\frac{3}{4}\right)\left(\frac{1}{2}z\right)^4 \text{ radians}. \quad . \quad . \quad . \quad (32)$$

3. *Tabular Values of* $N_0(z)$, $N_1(z)$, $\phi_0(z)$, *and* $\phi_1(z)$.

By aid of the preceding formulæ and the values of ker z, kei z, ker$'z$, and kei$'z$ given in the Brit. Assoc. Report for 1915, these functions have been computed for $z=0$ to 10, using seven figure logarithms. The tables are given in the Appendix at the end of this paper.

4. *Eddy Current Loss in Cylindrical Tube situated in Alternating Magnetic Field.*

The tube is placed coaxially in a solenoid. To avoid complicating the analysis, the ratio length to diameter in each case is assumed to be large enough to make the magnetic field in the tube wholly axial, and the loss will be evaluated for the tube alone. This case occurs in induction furnaces of the eddy current type with a tubular charge. As a particular case we can cite the expulsion of gas from the electrodes of a thermionic valve placed in a solenoid carrying a radio frequency current whilst evacuation is taking place. The circuit conditions are assumed to be such that the current is the same throughout the solenoid.

It can be shown * that the magnetizing force at a radius r is

$$\mathrm{H}=\mathrm{A}_1\mathrm{J}_0(mri^{3/2})+\mathrm{B}_1\mathrm{K}_0(mri^{1/2}), \qquad (33)$$

where $m^2=4\pi\mu\omega/\rho$, μ=permeability, ρ=resistivity, $\omega/2\pi$ =frequency.

When $r=a$, the outer radius, let $\mathrm{H}=\mathrm{H}_0$, and we get

$$\mathrm{H}_0=\mathrm{A}_1\mathrm{J}_0(mai^{3/2})+\mathrm{B}_1\mathrm{K}_0(mai^{1/2}). \qquad (34)$$

At the inner radius $r=b$, $d\mathrm{H}/dr=0$, so from (33)

$$\mathrm{A}_1mi^{3/2}\mathrm{J}_1(mbi^{3/2})+\mathrm{B}_1mi^{1/2}\mathrm{K}_1(mbi^{1/2})=0. \qquad (35)$$

From (33), (34), and (35) we obtain

$$\mathrm{H}=\mathrm{H}_0\left[\frac{\mathrm{K}_0(mri^{1/2})\mathrm{J}_1(mbi^{3/2})+i\mathrm{J}_0(mri^{3/2})\mathrm{K}_1(mbi^{1/2})}{\mathrm{K}_0(mai^{1/2})\mathrm{J}_1(mbi^{3/2})+i\mathrm{J}_0(mai^{3/2})\mathrm{K}_1(mbi^{1/2})}\right]. \qquad (36)$$

The field within a ring 1 cm. long, of radius r and thickness dr, due to the current $d\mathrm{I}=\sigma\,dr$ (where σ is the current density), is $-d\mathrm{H}=4\pi\sigma dr$, so

$$\sigma=-\frac{1}{4\pi}\frac{d\mathrm{H}}{dr}. \qquad (37)$$

Differentiating (36) we obtain

$$\sigma=\frac{\mathrm{H}_0mi^{1/2}}{4\pi}\left[\frac{\mathrm{K}_1(mri^{1/2})\mathrm{J}_1(mbi^{3/2})-\mathrm{J}_1(mri^{3/2})\mathrm{K}_1(mbi^{1/2})}{\mathrm{K}_0(mai^{1/2})\mathrm{J}_1(mbi^{3/2})+i\mathrm{J}_0(mai^{3/2})\mathrm{K}_1(mbi^{1/2})}\right] \qquad (38)$$

$$=\frac{\mathrm{H}_0mi^{1/2}}{4\pi}\left[\frac{\mathrm{S}}{\mathrm{Q}}\right], \qquad (39)$$

* A. Russell, 'Alternating Currents,' vol. i. pp. 209, 503, 2nd ed. (1914). Also 'Bessel Functions for Engineers,' p. 143.

where S and Q represent respectively the numerator and denominator of (38) and H_0 is a root mean square value.

The loss (heat generated) in the ring is

$$\mathrm{R}d\mathrm{I}^2=\left(\frac{2\pi r\rho}{dr}\right)\sigma^2\,dr^2$$

$$=2\pi\rho\sigma^2 r\,dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The loss per cm. length of tube is, therefore,

$$\mathrm{P}=2\pi\rho\int_b^a \sigma^2 r\,dr,$$

$$=\frac{\mathrm{H}_0{}^2 m^2\rho}{8\pi|\mathrm{Q}|^2}\int_b^a |\mathrm{S}|^2 r\,dr. \quad . \quad . \quad . \quad . \quad . \quad (41)$$

We have now to find $|S|^2$ in (41). From (38) we get

$$\mathrm{S}=[\mathrm{M}_1(b)\mathrm{N}_1(r)e^{i\{\theta_1(b)+\phi_1(r)\}}-\mathrm{M}_1(r)\mathrm{N}_1(b)e^{i\{\theta_1(r)+\phi_1(b)\}}]e^{\frac{1}{2}\pi i}, \quad . \quad . \quad . \quad . \quad (42)$$

where (b) is written for (mb) and (r) for (mr); so

$$|\mathrm{S}|^2=\mathrm{M}_1{}^2(b)\mathrm{N}_1{}^2(r)+\mathrm{M}_1{}^2(r)\mathrm{N}_1{}^2(b)-2\mathrm{M}_1(b)\mathrm{N}_1(b)\mathrm{M}_1(r)\mathrm{N}_1(r)$$
$$\cos\{(\phi_1-\theta_1)_r+(\theta_1-\phi_1)_b\}. \quad . \quad . \quad . \quad (43)$$

Putting the first two terms of (43) in the integrand of (41), and omitting the external factor, after integrating and inserting the limits, we obtain

$$\frac{\mathrm{N}_1{}^2(b)}{m}[a\mathrm{M}_0(a)\mathrm{M}_1(a)\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)_a$$
$$-b\mathrm{M}_0(b)\mathrm{M}_1(b)\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)_b]$$
$$+\frac{\mathrm{M}_1{}^2(b)}{m}[a\mathrm{N}_0(a)\mathrm{N}_1(a)\cos(\phi_0-\phi_1+\tfrac{1}{4}\pi)_a$$
$$-b\mathrm{N}_0(b)\mathrm{N}_1(b)\cos(\phi_0-\phi_1+\tfrac{1}{4}\pi)_b]. \quad (44)$$

The third term in (43) can be written

$$-\mathrm{M}_1(b)\mathrm{N}_1(b)\mathrm{M}_1(r)\mathrm{N}_1(r)[e^{i\{(\phi_1-\theta_1)_r+(\theta_1-\phi_1)_b\}}$$
$$+e^{-i\{(\phi_1-\theta_1)_r+(\theta_1-\phi_1)_b\}}]$$
$$=\mathrm{M}_1(b)\mathrm{N}_1(b)[e^{i\{(\theta_1-\phi_1)_b+\frac{1}{2}\pi\}}\mathrm{J}_1(mri^{-3/2})\mathrm{K}_1(mri^{1/2})$$
$$+e^{-i\{(\theta_1-\phi_1)_b+\frac{1}{2}\pi\}}\mathrm{J}_1(mri^{3/2})\mathrm{K}_1(mri^{-1/2})]. \quad (45)$$

On inserting (45) in (41) we obtain two integrals of the type *

$$\int J_1(kr)K_1(lr)r\,dr = \frac{r}{k^2+l^2}\{lJ_1(kr)K_1'(lr) - kK_1(lr)J_1'(kr)\}. \quad . \quad . \quad . \quad (46)$$

Taking the separate terms in (45) we have, on omission of the constant factors,

$$\int J_1(mri^{-3/2})K_1(mri^{1/2})r\,dr = \frac{r}{2m}\{J_1(mri^{-3/2})K_1'(mri^{1/2}) + K_1(mri^{1/2})J_1'(mri^{-3/2})\}e^{-\frac{1}{4}\pi i}, \quad (47)$$

$$\int J_1(mri^{3/2})K_1(mri^{-1/2})r\,dr = \frac{r}{2m}\{J_1(mri^{3/2})K_1'(mri^{-1/2}) + K_1(mri^{-1/2})J_1'(mri^{3/2})\}e^{\frac{1}{4}\pi i}. \quad . \quad (48)$$

Now

$$J_1'(mri^{-3/2}) = -\frac{J_1(mri^{-3/2})}{mri^{-3/2}} + J_0(mri^{-3/2})$$
$$= -\frac{M_1(r)}{mr}e^{-i\{\theta_1(r)-\frac{3}{4}\pi\}} + M_0(r)e^{-i\theta_0(r)}. \quad (49)$$

Similarly

$$J_1'(mri^{3/2}) = -\frac{M_1(r)}{mr}e^{i\{\theta_1(r)-\frac{3}{4}\pi\}} + M_0(r)e^{i\theta_0(r)}, \quad . \quad (50)$$

$$K_1'(mri^{-1/2}) = -\frac{N_1(r)}{mr}e^{-i\{\phi_1(r)+\frac{1}{4}\pi\}} - N_0(r)e^{-i\phi_0(r)}, \quad (51)$$

$$K_1'(mri^{1/2}) = -\frac{N_1(r)}{mr}e^{i\{\phi_1(r)+\frac{1}{4}\pi\}} - N_0(r)e^{i\phi_0(r)}. \quad . \quad (52)$$

Substituting from (49) and (52) in (47) we obtain

$$\int J_1(mri^{-3/2})K_1(mri^{1/2})r\,dr = \frac{r}{2m}[M_0(r)N_1(r)e^{i(\phi_1-\theta_0+\frac{1}{4}\pi)_r} - M_1(r)N_0(r)e^{-i(\theta_1-\phi_0+\frac{1}{4}\pi)_r}]. \quad (53)$$

Substituting from (50) and (51) in (48) we obtain

$$\int J_1(mri^{3/2})K_1(mri^{-1/2})r\,dr = \frac{r}{2m}[M_0(r)N_1(r)e^{-i(\phi_1-\theta_0+\frac{1}{4}\pi)_r} - M_1(r)N_0(r)e^{i(\theta_1-\phi_0+\frac{1}{4}\pi)_r}]. \quad (54)$$

* See 'Bessel Functions for Engineers,' p. 115.

Inserting the factors $M_1(b)N_1(b)e^{\pm i\{(\theta_1-\phi_1)_b+\frac{1}{2}\pi\}}$ from (45), and adding (53), (54) thereafter, we find that the value of the indefinite integral of the third term of (43), used as the integrand of (41), is

$$\frac{r}{m}M_1(b)N_1(b)[M_0(r)N_1(r)\cos\{(\phi_1-\theta_0)_r+(\theta_1-\phi_1)_b+\tfrac{3}{4}\pi\} -M_1(r)N_0(r)\cos\{(\theta_1-\phi_0)_r-(\theta_1-\phi_1)_b-\tfrac{1}{4}\pi\}]. \quad (55)$$

Inserting the limits a and b in (55), the value of the integral in (41) corresponding to the third term of (43) is

$$M_1(b)N_1(b)\left\{\begin{array}{l}\frac{a}{m}[M_0(a)N_1(a)\cos\{(\phi_1-\theta_0)_a+(\theta_1-\phi_1)_b+\tfrac{3}{4}\pi\} \\ \qquad -M_1(a)N_0(a)\cos\{(\theta_1-\phi_0)_a \\ \qquad -(\theta_1-\phi_1)_b-\tfrac{1}{4}\pi\}] \\ -\frac{b}{m}[M_0(b)N_1(b)\cos(\theta_1-\theta_0+\tfrac{3}{4}\pi)_b \\ \qquad -M_1(b)N_0(b)\cos(\phi_1-\phi_0-\tfrac{1}{4}\pi)_b]\end{array}\right\}. \quad (56)$$

Adding (44) and (56), and introducing the factors outside the integral (41), we find that the loss per cm. length of the tube is

$$\frac{H_0^2\rho ma\left\{\begin{array}{l}N_1^2(mb)M_0(ma)M_1(ma)\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)_{ma} \\ +M_1^2(mb)N_0(ma)N_1(ma)\cos(\phi_1-\phi_0-\tfrac{1}{4}\pi)_{ma} \\ -M_1(mb)N_1(mb)[M_0(ma)N_1(ma)\cos\{(\phi_1-\theta_0)_{ma}+(\theta_1-\phi_1)_{mb}-\tfrac{1}{4}\pi\} \\ \qquad +M_1(ma)N_0(ma)\cos\{\theta_1-\phi_0)_{ma}-(\theta_1-\phi_1)_{mb}-\tfrac{1}{4}\pi\}]\end{array}\right\}}{8\pi[M_1^2(mb)N_0^2(ma)+M_0^2(ma)N_1^2(mb)- -2M_0(ma)M_1(mb)N_0(ma)N_1(mb)\cos\{(\phi_0-\theta_0)_{ma}+(\theta_1-\phi_1)_{mb}\}]} \quad (57)$$

When ma and mb are large enough the asymptotic expansions take the approximate forms

$$M_0(z)=M_1(z)=e^{z/\sqrt{2}}/\sqrt{(2\pi z)}\,; \quad (58)$$

$$\theta_0(z)=\frac{z}{\sqrt{2}}-\frac{1}{8}\pi\,;\quad \theta_1(z)=\frac{z}{\sqrt{2}}+\frac{3}{8}\pi\,; \quad (59)$$

$$N_0(z)=N_1(z)=\sqrt{\left(\frac{\pi}{2z}\right)}e^{-z/\sqrt{2}}\,; \quad (60)$$

$$\phi_0(z)=-\frac{z}{\sqrt{2}}-\frac{1}{8}\pi\,;\quad \phi_1(z)=-\frac{z}{\sqrt{2}}-\frac{5}{8}\pi. \quad (61)$$

If (58)–(61) are substituted in (57) we obtain the simple form

$$P=\frac{H_0^2\rho ma}{8\sqrt{2}\pi}\left[\frac{\sinh x-\sin x}{\cosh x+\cos x}\right], \quad . \quad . \quad . \quad . \quad (62)$$

where $x=\sqrt{2}m(a-b)$ and $mb>10$.

When $x>4$ (62) can be written

$$P=H_0^2\rho ma/8\sqrt{2}\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

this being identical with the loss in a solid rod when $ma>10$.

5. *Effective Resistance due to Core* (R_e).

The loss per cm.³ is

$$P_{c.c.}=\frac{H_0^2\rho ma}{8\pi^2(a^2-b^2)}\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where ψ is the fraction in (57). Substituting $H_0=\frac{4\pi nI_0}{10l}$, the total loss in a volume V of the tube is

$$P_v=\frac{2n^2\rho maI_0^2\psi V}{l^2(a^2-b^2)}\times 10^{-9} \text{ watts}, \quad . \quad . \quad . \quad (65)$$

I_0 being the r.m.s. value of the solenoid current in amperes.

Since $P_v=I_0^2R_e$, it follows that

$$R_e=\frac{2n^2\rho ma\psi V}{l^2(a^2-b^2)}\times 10^{-8} \text{ ohms}. \quad . \quad . \quad . \quad (66)$$

When ma and mb are large enough the value of ψ can be taken from (62), so

$$R_e=\frac{\sqrt{2}n^2\rho maV}{l^2(a^2-b^2)}\left[\frac{\sinh x-\sin x}{\cosh x+\cos x}\right]. \quad . \quad . \quad . \quad (67)$$

6. *Effective Inductance of Toroid with Tubular Core* (L_e).

The magnetic flux inside the metal section of the tube is

$$\Phi=2\pi\mu\int_b^a Hr\,dr$$

$$=\frac{2\pi\mu H_0}{mQi^{1/2}}\left\{r\left[-J_1(mbi^{3/2})K_1(mri^{1/2}) + K_1(mbi^{1/2})J_1(mri^{3/2})\right]\right\}_b^a$$

$$\Phi = \frac{2\pi\mu H_0 a}{m}\left\{ e^{\frac{1}{4}\pi i}\left[\frac{-M_1(b)N_1(a)e^{i\{\theta_1(b)+\phi_1(a)\}} + M_1(a)N_1(b)e^{i\{\theta_1(a)+\phi_1(b)\}}}{M_1(b)N_0(a)e^{i\{\theta_1(b)+\phi_0(a)\}} - M_0(a)N_1(b)e^{i\{\theta_0(a)+\phi_1(b)\}}}\right]\right\} \quad (68)$$

The modulus of the bracketed quantity in (68) is

$$Z = \left[\frac{M_1^2(b)N_1^2(a) + M_1^2(a)N_1^2(b) - 2M_1(b)N_1(b)M_1(a)N_1(a)\cos\{(\theta_1-\phi_1)_a-(\theta_1-\phi_1)_b\}}{M_1^2(b)N_0^2(a) + M_0^2(a)N_1^2(b) - 2M_0(a)N_0(a)M_1(b)N_1(b)\cos\{(\phi_0-\theta_0)_a-(\phi_1-\theta_1)_b\}}\right]^{\frac{1}{2}} \quad (69)$$

Thus the total flux in the cross-section of the tube is

$$\Phi = \frac{2\pi\mu H_0 a Z}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

For a toroid ($d_m >> 2a$)

$$H_0 = \frac{4\pi n I_0}{\pi d_m} = \frac{4nI_0}{d_m}, \quad . \quad . \quad . \quad . \quad . \quad (71)$$

where d_m is the mean diameter. Substituting this value o. H_0 in (70) we obtain the flux interlinkage

$$\Phi n = \frac{8\pi\mu n^2 a Z I_0}{m d_m},$$

and since the inductance is Φn per unit current,

$$L_e = \frac{8\pi\mu n^2 a Z \times 10^{-9}}{m d_m} \text{ henry.} \quad . \quad . \quad . \quad (72)$$

When ma and mb are large enough for the asymptotic formulæ to be used, (72) becomes

$$L_e = \frac{8\pi n^2 \mu a}{m d_m}\left\{\frac{\cosh x - \cos x}{\cosh x + \cos x}\right\}^{1/2} \quad . \quad . \quad . \quad (73)$$

$$= \frac{8\pi n^2 \mu a}{m d_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

when $x > 4$. When $b \to 0$, (74) becomes

$$L_e = \frac{4\pi n^2 A}{l}\left(\frac{2\mu}{ma}\right) = \frac{4\pi n^2 A}{l}\mu_e, \quad . \quad . \quad . \quad . \quad (75)$$

this being the effective inductance of a toroid with a solid core of area A, radius a, and effective permeability μ_e.

The above formulæ include the flux in the metallic cross-section of the tube only. In addition (apart from that external to the tube) there is the flux in the cavity of radius b. This is $\pi b^2 H_b$, where H_b is the value of (36) when $r=b$. Thus

$$\pi b^2 H_b = \pi b^2 H_0 \left[\frac{M_1(b)N_0(b)e^{i\{\theta_1(b)+\phi_0(b)\}} - M_0(b)N_1(b)e^{i\{\theta_0(b)+\phi_1(b)\}}}{M_1(b)N_0(a)e^{i\{\theta_1(b)+\phi_0(a)\}} - M_0(a)N_1(b)e^{i\{\theta_0(a)+\phi_1(b)\}}}\right]. \quad (76)$$

To get the inductance due to the whole flux associated with the *tube* it is necessary to add the contributions from (68) and (76), and proceed as shown above. When x is large enough $H_b \to 0$, so L_e is given by either (73) or (74).

The ratio of H at radius b to that at radius a is a measure of the efficacy of the tube as a magnetic screen. This is

$$\phi = \left[\frac{M_1(b)N_0(b)e^{i\{\theta_1(b)+\phi_0(b)\}} - M_0(b)N_1(b)\,e^{i\{\theta_0(b)+\phi_1(b)\}}}{M_1(b)N_0(a)e^{i\{\theta_1(b)+\phi_0(a)\}} - M_0(a)N_1(b)e^{i\{\theta_0(a)+\phi_1(b)\}}}\right]. \quad . \quad . \quad . \quad (77)$$

When x is large enough $\phi \to 0$, so the screening is almost complete *.

7. *Impedance of Long Cylindrical Tube.*

Finally, it may be of interest to transform the usual formula for the impedance per cm. length of a cylindrical tube of outer and inner radii a, b to its polar form. The cartesian form † is

$$R_e + i\omega L_e = \frac{ipm}{2\pi a}\left[\frac{(\mathrm{ber}\,ma + i\,\mathrm{bei}\,ma)(\mathrm{ker}'mb + i\,\mathrm{kei}'mb) - (\mathrm{ber}'mb + i\,\mathrm{bei}'mb)(\mathrm{ker}\,ma + i\,\mathrm{kei}\,ma)}{(\mathrm{ber}'ma + i\,\mathrm{bei}'ma)(\mathrm{ker}'mb + i\,\mathrm{kei}'mb) - (\mathrm{ber}'mb + i\,\mathrm{bei}'mb)(\mathrm{ker}'ma + i\,\mathrm{kei}'ma)}\right]. \quad . \quad . \quad (78)$$

* The value of ϕ is then $\sqrt{a/b}/\cosh \frac{1}{2}x(1+i)$. In the case of a thin tube $(a-b)=t$, $a/b \doteq 1$, and if x is large enough $|\phi| \doteq 2e^{-mt/\sqrt{2}}$.

† O. Heaviside. 'Electrician,' xvii. p. 252 (1886); 'Electrical Papers,' i. p. 69; H. B. Dwight, Jour. Am. I. E. E. xlii. p. 827 (1923).

The polar form of (78) is

$$R_e+i\omega L_e=\frac{\rho m}{2\pi a}\left[\frac{M_1(mb)N_0(ma)e^{i\{\theta_1(mb)+\phi_0(ma)-\frac{1}{4}\pi\}}-M_0(ma)N_1(mb)e^{i\{\theta_0(ma)+\phi_1(mb)-\frac{1}{4}\pi\}}}{M_1(ma)N_1(mb)e^{i\{\theta_1(ma)+\phi_1(mb)\}}-M_1(mb)N_1(ma)e^{i\{\theta_1(mb)+\phi_1(ma)\}}}\right]. \quad . \quad . \quad . \quad (79)$$

When $b\to 0$ we have the case of a solid wire, and (79) becomes*

$$R_e+i\omega L_e=\left(\frac{\rho m}{2\pi a}\right)\frac{M_0(ma)}{M_1(ma)}e^{i\{\theta_0(ma)-\theta_1(ma)+\frac{3}{4}\pi\}}. \quad (80)$$

If ma and mb are large enough for formulæ (58)–(61) to be used, (79) reduces to the well-known forms obtained by Russell †, namely

$$R_e=\frac{\rho m}{2\sqrt{2}\pi a}\left[\frac{\sinh x+\sin x}{\cosh x-\cos x}\right]; \quad . \quad . \quad . \quad (81)$$

$$L_e=\frac{\sqrt{2}\mu}{ma}\left[\frac{\sinh x-\sin x}{\cosh x+\cos x}\right]. \quad . \quad . \quad . \quad (82)$$

Under the same condition (80) gives

$$R_e=\frac{\rho m}{2\sqrt{2}\pi a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

$$L_e=\frac{\sqrt{2}\mu}{ma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

these being the values of (81) and (82) when $x=\sqrt{2}m(a-b)$ is large enough. The above values of L_e are due solely to the magnetic flux within the metal.

Appendix.

z.	$N_0(z)$.	$\phi_0(z)$ (degrees).	$N_1(z)$.	$\phi_1(z)$ (degrees).
0·0 ..	∞	— 0·000	∞	225·000
0·1 ..	2·542 1	17·794	9·962 0	224·160
0·2 ..	1·891 7	23·626	4·928 0	222·407
0·3 ..	1·525 0	28·733	3·231 5	220·131
0·4 ..	1·274 6	33·517	2·372 3	217·503

* ‘Bessel Functions for Engineers,’ p. 140. $M_0(z)$, $\theta_0(z)$, $M_1(z)$, and $\theta_1(z)$ are tabulated therein.

† Proc. Phys. Soc. xxi. p. 581 (1909); Phil. Mag. xvii. p. 524 (1909).

z.	$N_0(z)$.	$\phi_0(z)$ (degrees).	$N_1(z)$.	$\phi_1(z)$ (degrees).
0·5	1·087 94	38·119	1·850 1	214·624
0·6	0·941 68	42·604	1·497 6	211·557
0·7	0·823 26	47·008	1·243 10	208·344
0·8	0·725 17	51·353	1·050 59	205·016
0·9	0·642 54	55·654	0·899 90	201·597
1·0	0·572 03	59·920	0·778 87	198·101
1·1	0·511 25	−64·159	0·679 69	194·544
1·2	0·458 43	68·375	0·597 11	190·934
1·3	0·412 21	72·573	0·527 47	187·279
1·4	0·371 55	76·755	0·468 10	183·587
1·5	0·335 60	80·925	0·417 05	179·861
1·6	0·303 68	85·083	0·372 81	176·108
1·7	0·275 25	89·232	0·334 24	172·329
1·8	0·249 85	93·372	0·300 43	168·529
1·9	0·227 09	97·505	0·270 64	164·709
2·0	0·206 64	101·632	0·244 29	160·872
2·1	0·188 24	−105·753	0·220 90	157·020
2·2	0·171 65	109·869	0·200 07	153·154
2·3	0·156 66	113·981	0·181 47	149·276
2·4	0·143 09	118·088	0·164 81	145·386
2·5	0·130 81	122·192	0·149 85	141·487
2·6	0·119 66	126·293	0·136 41	137·578
2·7	0·109 53	130·390	0·124 29	133·661
2·8	0·100 319	134·485	0·113 353	129·736
2·9	0·091 936	138·578	0·103 466	125·805
3·0	0·084 299	142·668	0·094 515	121·866
3·1	0·077 335	−146·756	0·086 402	117·922
3·2	0·070 979	150·842	0·079 039	113·972
3·3	0·065 175	154·927	0·072 348	110·018
3·4	0·059 870	159·009	0·066 264	106·058
3·5	0·055 018	163·090	0·060 724	102·095
3·6	0·050 578	167·170	0·055 677	98·127
3·7	0·046 513	171·249	0·051 074	94·155
3·8	0·042 789	175·326	0·046 873	90·180
3·9	0·039 376	179·402	0·043 036	86·201
4·0	0·036 246	183·477	0·039 530	82·220
4·1	0·033 374	−187·551	0·036 323	78·235
4·2	0·030 738	191·624	0·033 389	74·248
4·3	0·028 318	195·697	0·030 703	70·258
4·4	0·026 095	199·768	0·028 242	66·266
4·5	0·024 052	203·839	0·025 986	62·272
4·6	0·022 174	207·909	0·023 918	58·275
4·7	0·020 447	211·978	0·022 021	54·276
4·8	0·018 859	216·047	0·020 280	50·276
4·9	0·017 398	220·115	0·018 681	46·274
5·0	0·016 052	224·182	0·017 213	42·270
5·1	0·014 814	−228·249	0·015 864	38·264
5·2	0·013 674	232·315	0·014 624	34·257
5·3	0·012 624	236·381	0·013 484	30·248

z.	$N_0(z)$.	$\phi_0(z)$ (degrees).	$N_1(z)$.	$\phi_1(z)$ (degrees).
5·4 ..	0·011 656	240·446	0·012 435	26·238
5·5 ..	0·010 764 0	244·511	0·011 470 6	22·226
5·6 ..	0·009 942 0	248·575	0·010 582 8	18·214
5·7 ..	0·009 184 1	252·639	0·009 765 5	14·200
5·8 ..	0·008 485 2	256·703	0·009 013 0	10·185
5·9 ..	0·007 840 6	260·766	0·008 320 0	6·169
6·0 ..	0·007 246 0	264·829	0·007 681 5	+2·151
6·1 ..	0·006 697 3	−268·892	0·007 093 2	−1·867
6·2 ..	0·006 191 0	272·954	0·006 550 9	5·886
6·3 ..	0·005 723 6	277·016	0·006 051 1	9·906
6·4 ..	0·005 292 2	281·078	0·005 590 2	13·927
6·5 ..	0·004 893 8	285·139	0·005 165 1	17·949
6·6 ..	0·004 526 0	289·200	0·004 773 0	21·972
6·7 ..	0·004 186 2	293·261	0·004 411 2	25·995
6·8 ..	0·003 872 4	297·321	0·004 077 4	30·019
6·9 ..	0·003 582 4	301·382	0·003 769 4	34·044
7·0 ..	0·003 314 5	305·442	0·003 485 0	38·070
7·1 ..	0·003 067 0	−309·502	0·003 222 4	−42·096
7·2 ..	0·002 838 2	313·561	0·002 980 0	46·123
7·3 ..	0·002 626 7	317·621	0·002 756 1	50·150
7·4 ..	0·002 431 1	321·680	0·002 549 3	54·178
7·5 ..	0·002 250 4	325·739	0·002 358 3	58·207
7·6 ..	0·002 083 2	329·798	0·002 181 8	62·236
7·7 ..	0·001 928 6	333·857	0·002 018 7	66·266
7·8 ..	0·001 785 7	337·915	0·001 868 0	70·296
7·9 ..	0·001 653 4	341·974	0·001 728 7	74·327
8·0 ..	0·001 531 1	346·032	0·001 599 9	78·358
8·1 ..	0·001 417 9	−350·090	0·001 480 8	−82·389
8·2 ..	0·001 313 2	354·148	0·001 370 8	86·421
8·3 ..	0·001 216 3	358·205	0·001 269 0	90·454
8·4 ..	0·001 126 68	362·263	0·001 174 85	94·487
8·5 ..	0·001 043 69	366·321	0·001 087 79	98·520
8·6 ..	0·000 966 89	370·378	0·001 007 26	102·554
8·7 ..	0·000 895 79	374·435	0·000 932 76	106·588
8·8 ..	0·000 829 97	378·492	0·000 863 83	110·623
8·9 ..	0·000 769 04	382·549	0·000 800 06	114·657
9·0 ..	0·000 712 63	386·606	0·000 741 04	118·693
9·1 ..	0·000 660 39	−390·663	0·000 686 43	−122·728
9·2 ..	0·000 612 02	394·720	0·000 635 88	126·764
9·3 ..	0·000 567 22	398·776	0·000 589 10	130·800
9·4 ..	0·000 525 73	402·832	0·000 545 79	134·836
9·5 ..	0·000 487 30	406·889	0·000 505 70	138·873
9·6 ..	0·000 451 71	410·945	0·000 468 58	142·910
9·7 ..	0·000 418 73	415·001	0·000 434 21	146·947
9·8 ..	0·000 388 19	419·057	0·000 402 39	150·985
9·9 ..	0·000 359 89	423·113	0·000 372 92	155·023
10·0 ..	0·000 333 67	427·169	0·000 345 63	159·061

Dr. L. J. Comrie, Secretary to the British Association Committee on Mathematical Tables, has very kindly checked the above in their original form as computed by seven-figure logarithms. In no case will the error in the fifth significant figure exceed one unit.

LXII. *On the Solution of Boundary Problems in Mathematical Physics.* By J. Neufeld *.

THE author of the above paper (see Phil. Mag. ser. 7, xvii. p. 987 (1934)) notifies the following corrections which have been brought to his attention by Mr. C. C. Chambers :—

1. Page 989, equation (6):

$$\int_k^l \int_{\phi_1(y_1)}^{\phi_2(y_1)} \frac{\partial^2 H}{\partial x^2} \epsilon^{-i(mx+ny)} dx\, dy = \int_k^l \epsilon^{-iny} \left[\frac{\partial H}{\partial x} \epsilon^{-imx}\right]_{x=\phi_1(y_1)}^{x=\phi_2(y_1)} dy$$

$$+ im \int_k^l \int_{\phi_1(y_1)}^{\phi_2(y_1)} \frac{\partial H}{\partial x} \epsilon^{-i(mx+ny)} dx\, dy$$

$$= \int_k^l \epsilon^{-iny} \left[\frac{\partial H}{\partial x} \epsilon^{imx}\right]_{x=\phi_1(y_1)}^{x=\phi_2(y_1)} dy + im \int_k^l \epsilon^{-iny} \left[H\epsilon^{-imx}\right]_{x=\phi_1(y_1)}^{x-\phi_2(y_1)} dx$$

$$-m^2 \int_k^l \int_{\phi_1(y_1)}^{\phi_2(y_1)} H\epsilon^{-i(mx+ny)} dx\, dy.$$

2. Equation (7) and namely :

$$[H]_{x=\phi_1(y_1)} = [H]_{x=\phi_2(y_1)}$$

is valid only in the case in which

$$\phi_1(y_1) = k_1 ; \quad \phi_2(y_1) = k_2,$$

where k_1, k_2 are constants. Consequently the scope of this article is limited to rectangular boundaries.

* Communicated by the Author.

LXIII. *Sources of various kinds near a Plane Boundary separating two different Media. By* GEORGE GREEN, *D.Sc., Lecturer on Physics in the Applied Physics Department, The University, Glasgow* *.

PROBLEMS in various subjects involving boundaries between media differing in properties present considerable difficulty in themselves; but the difficulty is largely due to the fact that very few standard or fundamental cases have been studied, involving transmission of waves across a boundary, to enable us to become familiar with the functions required and their interrelations, and the separate effects each gives rise to in the transition from one medium to another. This field of investigation presents opportunities to the pure mathematician, but it is from the standpoint of mathematical physics that the present investigation is being undertaken. In several previous papers † the author has developed a general method of dealing with problems involving transmission of effects across a boundary, the method being illustrated by the solution of problems in heat conduction. Attention has also been drawn to the fact that in the solution of problems by means of wave-trains many analogies can be observed in many different subjects, and that for subjects included within the scope of the subject of wave-motion the treatment of problems involving certain specified boundaries is the same for all. In wave-motion, problems involving sources either periodic, instantaneous, or continuous, are fundamental. The intention of the present paper is therefore to make a preliminary investigation of the problems presented by sources of various kinds placed near the plane boundary of separation of two different media, and to examine generally the nature and variety of the sources themselves that arise in the present case and may require to be considered in more general cases.

Primarily the sources to be considered are periodic

* Communicated by the Author.

† Phil. Mag. iii. Suppl. (April 1927), I.; v. (April 1928), II.; ix. (Feb. 1930), III.; xii. Suppl. (Aug. 1931), IV. Certain investigations promised in the course of these papers have since been completed by J. Robertson, M.A., B.Sc., in Phil. Mag. ser. 7, xv. (May 1933), V.; ser. 7, xviii. (July 1934), VI.

heat sources, and the fundamental differential equations to be satisfied on either side of the plane boundary separating medium I. and medium II. are respectively

$$\frac{\partial v}{\partial t}=\kappa_1\nabla^2(v) \quad \text{and} \quad \frac{\partial v}{\partial t}=\kappa_2\nabla^2(v). \quad . \quad . \quad (1)$$

As we shall take e^{ikt} as the term involving the time t in all cases, these reduce to

$$(\nabla^2-h^2)v=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $h_1^2=\frac{ik}{\kappa_1}$ for medium I. and $h_2^2=\frac{ik}{\kappa_2}$ for medium II.

Inasmuch as the equation $\frac{d^2\phi}{dt^2}=\pm\mathrm{V}^2\nabla^2(\phi)$, under similar conditions with respect to time t, reduces to

$$(\nabla^2\pm h^2)\phi=0, \quad \text{where} \quad h^2=\pm\frac{k^2}{\mathrm{V}^2}, \quad . \quad . \quad (2')$$

it will be evident that corresponding to any one of the various solutions of (1) representing a source in heat there is to be found a similar form representing a source in any of the other subjects included within wave-motion, and that the results obtained with reference to heat problems provide a group of parallel results with reference to sound, elasticity, optics, or wave-motion generally. Analogous results arise in many subjects, but the analogy is in general more complete provided that the quantity corresponding to temperature in the case of heat is also a scalar quantity. The correspondence referred to is sufficiently maintained when the boundary conditions appropriate to each subject are taken into consideration. In the case of heat conduction, at the plane interface between medium I. and medium II., the conditions to be fulfilled are

$$v_1=v_2 \quad \text{and} \quad -\mathrm{K}_1\frac{dv_1}{dx}=-\mathrm{K}_2\frac{dv_2}{dx} \quad . \quad . \quad . \quad (3)$$

for a boundary plane parallel to the (y, z) plane. The particular methods of solution here employed are, however, applicable to any other boundary conditions that might arise.

Plane Source.—To illustrate the general method of procedure to obtain a solution of equations (1) and (3), we

may take first the case of a plane source of heat of strength qe^{ikt} situated in medium I. at the plane defined by $x \cos \phi + y \sin \phi = 0$, the plane $x=0$ being the common boundary of the two media. The flow of heat from such a source is one-dimensional ($\frac{1}{2}qe^{ikt}$ per unit area in each direction), and the two positive wave-trains represented by the real and imaginary parts of

$$v_1 = f_1 e^{-h_1(x \cos \phi + y \sin \phi)}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $$f_1 = \frac{qe^{ikt}}{\{2K_1 h_1\}} \quad \text{and} \quad h_1 = \sqrt{\frac{ik}{\kappa_1}}$$

fulfil the condition required by the source in medium I. These wave-trains issue from the source on one side of the plane $x \cos \phi + y \sin \phi = 0$, in a direction inclined to the axis of x at an angle ϕ, which is also the usual angle of incidence of the normal to the wave-trains on the boundary plane, $x=0$. The expression (4) satisfies the differential equation (1) but not the boundary conditions (3); and to complete the solution required we must assume that the above trains incident on the boundary plane $x=0$ give rise to reflected wave-trains in medium I. represented by

$$v_1' = Af_1 e^{-h_1(-x \cos \phi + y \sin \phi)}, \quad . \quad . \quad . \quad . \quad (5)$$

and to refracted wave-trains in medium II. represented by

$$v_2 = Bf_2 e^{-h_2(x \cos \phi' + y \sin \phi')}, \quad f_2 = \frac{qe^{ikt}}{(2K_2 h_2)}, \quad . \quad . \quad (6)$$

where A and B represent boundary coefficients for reflexion and refraction respectively. The condition that all three wave-trains represented by (4), (5), (6) above should have the same phase at the boundary $x=0$ for all values of y leads at once to the relation

$$h_1 \sin \phi = h_2 \sin \phi' \quad \text{or} \quad \frac{\sin \phi}{\sqrt{\kappa_1}} = \frac{\sin \phi'}{\sqrt{\kappa_2}} \quad . \quad . \quad (7)$$

The application of the boundary conditions expressed in equation (3) to the solutions represented by $v_1 + v_1'$ for medium I. and by v_2 for medium II. determines the values of A and B. These are found to be

$$A = \frac{\dfrac{\cot \phi}{\cot \phi'} - \dfrac{K_2}{K_1}}{\dfrac{\cot \phi}{\cot \phi'} + \dfrac{K_2}{K_1}}; \quad B = \frac{K_2 h_2}{K_1 h_1}(1+A) = \frac{2\dfrac{K_2 \cos \phi}{K_1 \cos \phi'}}{\dfrac{\cot \phi}{\cot \phi'} + \dfrac{K_2}{K_1}}; \quad (8)$$

and (4), (5), and (6) then give a solution of (1) to represent the effect of a plane periodic source of heat placed near a boundary between two different media.

The solutions obtained above present features which are almost identical with those found in the corresponding solutions in sound and optics, etc. *. Regarding the matter from the point of view of wave-motion, in the above incident, reflected and refracted wave-trains, the wave-velocity or phase-velocity is given by (k/h_1), (k/h_1), (k/h_2) respectively, and (7) above is equivalent to $\sin\phi : \sin\phi' = V_1 : V_2$, the well-known law of sines. When $\kappa_2 > \kappa_1$ medium II. is optically less dense than medium I. Since (7) holds in all cases, we have

$$\cos\phi' = \sqrt{1 - \frac{\kappa_2}{\kappa_1}\sin^2\phi},$$

and if $\sin\phi$ exceeds $\sqrt{\frac{\kappa_1}{\kappa_2}}$ it is clear that $\cos\phi'$ cannot have a real value. The above expressions still represent the solution of the problem, but they require to be modified in accordance with $\cos\phi' = \sqrt{-a_1^2}$ or $\cos\phi' = -ia_1$, where a_1 is a real quantity. We have here the analogue of total reflexion in optics; but the case of heat differs from that of sound or light in this, that when the critical angle is exceeded there is still a transmitted wave of the type

$$\cos\left\{kt + (a_1x - y\sin\phi')\sqrt{\frac{k}{2\kappa_2}}\right\}.$$

The modifications required in this case involve certain changes in A and B and also a change in the direction of advance of the transmitted wave. The matter requires more complete examination, especially where the possibilities of a return flow of heat from medium II. to medium I. fall to be considered.

When the source in medium I. is an instantaneous source of heat of strength q the solution can be obtained

* See for example Rayleigh's 'Sound,' ii. pp. 79–85, 427–8; J. J. Thomson's 'Recent Researches,' pp. 407–414; Bateman, 'Electrical and Optical Wave-motion,' pp. 73–75.

by integration with respect to k from that given above; thus,

$$\frac{1}{\pi}\int_0^\infty dk(v_1+v_1') \text{ for medium I., and}$$

$$\frac{1}{\pi}\int_0^\infty dk\, v_2 \text{ for medium II.}$$

The integrands appearing in v_1, v_1', and v_2 are all similar functions of k. The integrations required have been performed in an earlier paper *, and need not be given in full. The complete solution for the instantaneous source in the case of direct or normal incidence, involving exactly similar integrations, has also been given in full in the same paper.

Line Source.—The investigation of the corresponding problem, where the initial source is a linear periodic source of strength qe^{ikt} per unit length of a line in medium I. parallel to the plane common to the two media and distant a from it, proceeds along similar lines to that in the problem already treated. The equations $(\nabla^2 \pm h^2)\phi = 0$ have each two symmetrical source solutions represented by the real and imaginary parts of

$$\left.\begin{aligned} e^{ikt}\mathrm{D}_0(h\rho) &= \frac{2}{\pi}\, e^{ikt}\int_0^\infty e^{-ih\rho\cosh u}\,du \\ \text{and} \qquad e^{ikt}\mathrm{K}_0(h\rho) &= e^{ikt}\int_0^\infty e^{-h\rho\cosh u}\,du \end{aligned}\right\} \quad . \quad . \quad . \quad (9)$$

respectively, the sources being situated at the origin, $\rho = 0$. These solutions, which can be expressed in various other forms, with their derivatives, by integration and differentiation, provide the materials for the solution to our problem.

The procedure then is as follows. Take first the solution of (1) given by

$$v_1 = e^{ikt - i\beta_1 x - \sqrt{(\beta_1{}^2 + h_1{}^2)}\,y}, \quad . \quad . \quad . \quad . \quad (10)$$

which represents a wave-train diverging from the origin O, taken at the initial source in medium I. Combine with this in medium I. a reflected wave-train given by

$$v_1' = \mathrm{A}e^{ikt - \beta_1 x' i - \sqrt{(\beta_1{}^2 + h_1{}^2)}\,y}, \quad x' = (2a - x), \quad . \quad (11)$$

* Phil. Mag. iii. Suppl. (April 1927).

which represents a wave-train diverging from O′, the image point of O with reference to the common boundary plane for the two media. The corresponding refracted wave-train may then be expressed by

$$v_2 = \mathrm{B}e^{ikt - i\beta_2(x-\alpha) - \sqrt{(\beta_2^2+h_2^2)}y}; \quad . \quad . \quad . \quad (12)$$

which represents a wave-train diverging from a point $(x=\alpha,\ y=0)$ into medium II. Apply now the condition that, at boundary $x=a$, all three wave-trains have the same phase. This gives the two equations defining β_2 and α in terms of β_1,

$$\beta_2(a-\alpha)=\beta_1 a \quad \text{and} \quad \sqrt{\beta_2^2+h_2^2}=\sqrt{\beta_1^2+h_1^2}. \quad (13)$$

Thereafter, by means of the two boundary conditions (3) the values of the two boundary coefficients A and B can be obtained. These values are

$$\mathrm{A}=\frac{\dfrac{\beta_1}{\beta_2}-\dfrac{\mathrm{K}_2}{\mathrm{K}_1}}{\dfrac{\beta_1}{\beta_2}+\dfrac{\mathrm{K}_2}{\mathrm{K}_1}} \quad \text{and} \quad \mathrm{B}=\frac{2\dfrac{\beta_1}{\beta_2}}{\dfrac{\beta_1}{\beta_2}+\dfrac{\mathrm{K}_2}{\mathrm{K}_1}}, \quad . \quad . \quad (14)$$

and the solution corresponding to one incident wave-train is now completely determined by v_1+v_1' for medium I., with v_2 for medium II.

It is instructive to compare this solution with that already obtained for a wave-train issuing from a plane source. Observing that in the incident wave the phase is constant along the line.

$$i\beta_1 x+\sqrt{\beta_1^2+h_1^2}\,.\,y=\text{constant}, \quad . \quad . \quad . \quad (15)$$

and taking ϕ, as before, as the inclination to the axis of x of the normal to this line at point (x, y), we obtain

$$\cot\phi=\frac{i\beta_1}{\sqrt{\beta_1^2+h_1^2}}; \text{ and likewise } \cot\phi'=\frac{i\beta_2}{\sqrt{\beta_2^2+h_2^2}}. \quad (16)$$

It follows that $\dfrac{\sin\phi}{\sqrt{\kappa_1}}=\dfrac{\sin\phi'}{\sqrt{\kappa_2}}$, as before, and that the values obtained for A and B in the two cases are in complete agreement. Equations (14) are in fact identical with equations (8) when β_1, β_2 are expressed in terms of ϕ, ϕ', and the conditions maintaining when $\kappa_2>\kappa_1$ are similar in the two cases. In (8) $\mathrm{B}f_2/f_1$ corresponds with B in (14)

When the incident wave-system represents a periodic line source emitting qe^{ikt} heat units per unit length, its form, determined in a previous paper (II. p. 703), is given by

$$v_1 = \frac{qe^{ikt}}{2\pi K_1} \cdot K_0\{h_1\rho\}, \quad \ldots \ldots \ldots \quad (17)$$

which is also equivalent to

$$v_1 = \frac{qe^{ikt}}{2\pi K_1}\int_0^\infty \frac{e^{-py}\cos\beta_1 x}{p}\, d\beta_1, \quad p^2=\beta_1^2+h_1^2. \quad (17')$$

This solution can obviously be obtained from (10) by integration.

The corresponding reflected and refracted wave-systems are then

$$v_1' = \frac{qe^{ikt}}{2\pi K_1}\int_0^\infty \frac{Ae^{-py}\cos\beta_1 x'\, d\beta_1}{p}, \quad x'=(2a-x), \,. \quad (18)$$

and $v_2 = \frac{qe^{ikt}}{2\pi K_1}\int_0^\infty \frac{Be^{-py}\cos\beta_2(x-\alpha)d\beta_1}{p}$ respectively. (19)

The values of A and B are determined by (14), and the complete solution is given by v_1+v_1' for medium I. and by v_2 for medium II. It is worthy of note that the reflected wave-system issues from the line $x=2a$, which is the image in the plane of separation of the two media of the original line-source at O. It consists of a group of terms representing sources of the type $K_n(h_1\rho)\cos n\theta$ derivable from (17) by differentiation together with a group of terms similarly derivable from the fundamental solution obtained from (17′) by changing $\cos\beta_1 x'$ into $\sin\beta_1 x'$. The nature of the composite source existing at O′ and of that represented by (19) may be understood from the more complete discussion given below of the corresponding problem relating to point sources. A fuller examination of the integrals appearing in (17), (18), (19) above is desirable when time permits.

The solution of the problem in which the initial source at O is an instantaneous heat source q is obtained by integrating the above solution with respect to k, and a further integration with respect to t gives the solution required when the source at O is a continuous one. The integrations with respect to k for (17) have been effected in a previous paper (II. pp. 705–6). The integrations

involved in (18) and (19) and the subsequent integrations with respect to k can be carried out according to the method indicated later for the corresponding point-source problems. It may be possible to examine the properties of the above solution more fully later. With suitable interpretation of the various constants (17), (18), (19) represent a line source near a plane boundary separating two media in a great variety of subjects.

Point source.—Consider now the problem of determining the effect of a periodic point source of strength qe^{ikt} situated in medium I. at the origin, a point distant a from the plane boundary $x=a$ common to the two media. The fundamental source solutions of equation (2′) now required are

$$\sqrt{\frac{2h}{\pi}}\frac{G_{\frac{1}{2}}(-hr)}{r^{\frac{1}{2}}}=\frac{e^{-ihr}}{r} \quad \text{and} \quad \sqrt{\frac{2h}{\pi}}\frac{K_{\frac{1}{2}}(hr)}{r^{\frac{1}{2}}}=\frac{e^{-hr}}{r}, \qquad (20)$$

yielding again two symmetrical wave-trains issuing from the origin as source.

Following the procedure adopted in previous cases we consider first a wave-train of the type

$$v_1=e^{ikt-ip_1r}K_0(\lambda_1\rho), \quad \text{where } p_1{}^2=\lambda_1{}^2-h_1{}^2, \quad \text{and } \rho^2=y^2+z^2 \quad . \quad (21)$$

issuing from the origin and representing the incident wave-train in medium I. The corresponding reflected and refracted wave-trains are then given by

$$v_1'=Ae^{ikt-ip_1x'}K_0(\lambda_1\rho), \text{ where } x'=(2a-x) \text{ as before}, \quad (22)$$

and

$$v_2=Be^{ikt-ip_2(x-\alpha)}K_0(\lambda_1\rho), \quad p_2{}^2=\lambda_1{}^2-h_2{}^2. \quad . \quad . \quad . \quad (23)$$

The phase condition formerly referred to gives the relation

$$ap_1=(a-\alpha)p_2, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the boundary conditions (3) supply the information necessary for the determination of the boundary coefficients A and B. Their values are

$$A=\frac{\dfrac{p_1}{p_2}-\dfrac{K_2}{K_1}}{\dfrac{p_1}{p_2}+\dfrac{K_2}{K_1}} \ : \ B=\frac{2.\dfrac{p_1}{p_2}}{\dfrac{p_1}{p_2}+\dfrac{K_2}{K_1}}=A+1. \quad . \quad (25)$$

When we express $K_0(\lambda_1\rho)$ in the above by means of its well-known asymptotic expansion we find that in the incident wave-train a line of constant phase through point (x, ρ) is defined by

$$ip_1x+\lambda_1\rho=\text{constant.} \quad . \quad . \quad . \quad . \quad (26)$$

This again leads to

$$\cot\phi=\frac{ip_1}{\lambda_1};\ \sin\phi=\frac{\lambda_1}{h_1}:\ \text{and}\ \cot\phi'=\frac{ip_2}{\lambda_1};\ \sin\phi'=\frac{\lambda_1}{h_2} \quad (27)$$

for the rays in the notation formerly employed. It is now clear that the law of sines, $\frac{\sin\phi}{\sqrt{\kappa_1}}=\frac{\sin\phi'}{\sqrt{\kappa_2}}$, holds in all the cases of wave-transmission that we have considered, and that the coefficients A and B have likewise the same values throughout. Equations (8), (14), and (25) are in fact identical in form when expressed in terms of ϕ and ϕ', so that the law of sines and the values obtained for A and B as functions of ϕ and ϕ' are valid in all cases of wave motion. When the angle of incidence exceeds the critical value, $\sin\phi=\sqrt{\frac{\kappa_1}{\kappa_2}}$, the special features of the above solution are the same as in the former solutions.

In the special problem proposed, the incident wave-system being that from a source of strength qe^{ikt}, the required value of v_1 is known and may be expressed in several equivalent forms :

$$v_1=\frac{qe^{ikt}}{4\pi K_1}\cdot\frac{e^{-h_1r}}{r}=\frac{qe^{ikt}}{4\pi K_1}\cdot\sqrt{\frac{2h_1}{\pi}}\,.\,K_{\frac{1}{2}}(h_1r)/r^{\frac{1}{2}} \quad . \quad . \quad . \quad (28)$$

$$=\frac{qe^{ikt}}{4\pi K_1}\cdot\frac{2}{\pi}\int_0^\infty dp_1\cos xp_1K_0\{\rho\sqrt{p_1^2+h_1^2}\}=\frac{qe^{ikt}}{4\pi K_1}\cdot\frac{2}{\pi}C(x).$$

$$. \quad . \quad . \quad (28')$$

This last form can obviously be arrived at by integration of (21) above. Hence, with (28) as incident wave, the reflected wave to be added to it to complete the solution for medium I. is then given by

$$v_1'=\frac{qe^{ikt}}{4\pi K_1}\cdot\frac{2}{\pi}\int_0^\infty dp_1\,A\cos x'p_1K_0\{\rho\sqrt{p_1^2+h_1^2}\},$$

$$x'=(2a-x), \quad (29)$$

and the corresponding refracted wave representing the solution in medium II. is

$$v_2=\frac{qe^{ikt}}{4\pi K_1}\cdot\frac{2}{\pi}\int_0^\infty dp_1 B\cos((x-\alpha)p_2)K_0\{\rho\sqrt{p_1^2+h_1^2}\}, \quad (30)$$

where A, B, p_2, and α are defined by equations given above. This represents a solution applicable with suitable modifications in the meanings of the symbols to all the wave-motion subjects. It is to be remarked that while the incident wave represents a simple point source at the origin, the reflected wave represents a group of multiple point sources situated at the image point of the origin with reference to the plane boundary common to both media. The refracted wave similarly represents multiple sources distributed over a region of the x axis defined by

$$\alpha=a\left(1-\frac{p_2}{p_1}\right).$$

Incident Wave-system.—With a view to the examination of the properties of the integrals appearing above and their expression in terms of known functions consider first the integral (28′) representing the incident wave-system. By expanding cos xp, and then integrating term by term, we obtain

$$C(x)=\sum_{n=0}^{n=\infty}(-1)^n\frac{x^{2n}}{2n\,!}\cdot\frac{2^{n-\frac{1}{2}}\Gamma(n+\frac{1}{2})h_1^{n+\frac{1}{2}}}{\rho^{n+\frac{1}{2}}}K_{-(n+\frac{1}{2})}(h_1\rho). \quad (31)$$

Similarly the corresponding integral solution with sin xp replacing cos xp is

$$S(x)=\sum_{n=0}^{n=\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)\,!}\frac{2^n\Gamma(n+1)h_1^{n+1}}{\rho^{n+1}}K_{-(n+1)}(h_1\rho). \quad (32)$$

Since by a property of Bessel functions

$$\frac{h_1^{n+1}}{\rho^{n+1}}K_{n+1}(h_1\rho)=-\frac{1}{\rho}\frac{d}{d\rho}\left\{\frac{h_1^n}{\rho^n}K_n(h_1\rho)\right\}, \quad (33)$$

the successive terms in (31) and (32) are obtainable by differentiation from the first term of the series in each case. This enables us to evaluate readily (28) and in like manner (29) and (30), and facilitates the subsequent integrations with respect to k. By means of the two evaluations given in former papers (I. pp. 787, 794;

II. pp. 705–6), which are reproduced here only for the real parts of the integrals,

$$\int_0^\infty dk\, e^{ikt} h_1^{\frac{1}{2}} \mathrm{K}_{\frac{1}{2}}(h_1\rho)/\rho^{\frac{1}{2}} = \frac{\pi e^{-\frac{\rho^2}{4\kappa_1 t}}}{(8\kappa_1 t^3)^{\frac{1}{2}}}, \quad . \quad . \quad (34)$$

$$\int_0^\infty dk\, e^{ikt} \mathrm{K}_0(h_1\rho) = \frac{\pi e^{-\frac{\rho^2}{4\kappa_1 t}}}{t}, \quad . \quad . \quad . \quad . \quad (35)$$

it is possible to carry out for (28) the complete process represented by $\frac{1}{\pi}\int_0^\infty dk\, v_1$, which gives the value required for an instantaneous source at the origin; thereby we obtain the usual well-known result. It is easy to verify also that a further integration with respect to t' leads to the value $\frac{q}{4\pi \mathrm{K}_1 r}$ for a continuous source of strength q situated at the origin. The above discussion, however, shows that a similar symmetrical, continuous source solution is obtainable by applying the process indicated above to the imaginary parts of the original solutions. These may be of considerable importance and may be examined later, together with similar unsymmetrical source solutions obtained from the S integral corresponding to (28′).

Reflected and Refracted Wave-systems.—Returning now to (29), the integral representing the reflected wave-system, we may express the value of A in a series of ascending powers of p_1:

$$\mathrm{A} = -1 + 2\frac{\mathrm{K}_1}{\mathrm{K}_2}\cdot\frac{p_1}{d} - 2\frac{\mathrm{K}_1^2}{\mathrm{K}_2^2}\cdot\frac{p_1^2}{d^2} + \frac{\mathrm{K}_1}{\mathrm{K}_2}\cdot\frac{2\mathrm{K}_1^2 - \mathrm{K}_2^2}{\mathrm{K}_2^2}\cdot\frac{p_1^3}{d^3} + . \quad (36)$$

where
$$d^2 = ik\left(\frac{1}{\kappa_1} - \frac{1}{\kappa_2}\right). \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The series of integrals thus obtained in (29) consists of two groups—the alternate terms which are even or odd with respect to p_1. In the even group the terms are all derivable from the C(x') integral (28) by differentiation with respect to x', each being of the form $\frac{d^n}{dx'^n}\mathrm{C}(x')$. These terms represent multiple sources similar in type to

the initial source in medium I. Their sum may be expressed as a series of Bessel functions of the type

$$\sum_{0}^{\infty} A_n \frac{K_{n+\frac{1}{2}}(h_1 r')}{r'^{\frac{1}{2}}} P_n(\cos\theta) = \sum_{0}^{\infty} A_n T_n, \quad . \quad . \quad (38)$$

and the successive terms obey a serial relation indicated by

$$\frac{d}{dx'} T_n = \frac{-h_1}{2m+1} \{T_{n+1} + nT_n\}. \quad . \quad . \quad . \quad (39)^*$$

The terms of the odd group are likewise derivable by differentiation with respect to x' from an original periodic source solution represented by the $S(x')$ integral corresponding to (28). As Bessel functions they appear to be expressible in terms of the type K_n instead of $K_{n+\frac{1}{2}}$.

The expression of coefficients A, B, in a series of descending powers of p_1 is given by

$$A = \frac{K_1 - K_2}{K_1 + K_2} - \frac{K_1 K_2}{(K_1 + K_2)^2} \cdot \frac{d^2}{p_1^2} + \frac{K_1 K_2 (K_1 + 3K_2)}{4(K_1 + K_2)^3} \cdot \frac{d^4}{p_1^4} + \text{etc.} \quad . \quad . \quad . \quad (40)$$

$$B = \frac{2K_1}{K_1 + K_2} - \frac{K_1 K_2}{(K_1 + K_2)^2} \cdot \frac{d^2}{p_1^2} + \frac{K_1 K_2 (K_1 + 3K_2)}{4(K_1 + K_2)^3} \cdot \frac{d^4}{p_1^4} + \text{ete.} \quad . \quad . \quad . \quad (41)$$

These values, inserted into (29) and (30), provide new expressions for the reflected and refracted wave-systems. Each successive integral term so obtained from (29) is again a solution of (1), and each term is derivable from the fundamental $C(x')$ integral by two or more integrations with respect to x'. Considering the reflected wave-system (29), expressed by means of (40), its first term represents a source similar in type to the original source at O of strength $\frac{K_1 - K_2}{K_1 + K_2} q e^{ikt}$. The treatment of the second and later terms may be sufficiently illustrated by the following evaluation of the second integral. To pass from (29) to the corresponding instantaneous source

* A similar formula holds for the $I_{n+\frac{1}{2}}$ solutions, the minus being replaced by a plus.

solution in the case of the second term we require to evaluate T_2 defined by

$$T_2 = \frac{q}{2\pi^3 K_1}\int_0^\infty dk\, e^{ikt}\int_0^\infty dp\,\frac{\cos x'p}{p^2}K_0\{\rho\sqrt{p^2+h_1^2}\}. \quad (42)$$

To obtain T_1, the corresponding value for the first term, replace p^{-2} by unity.

By expanding the cosine, as in (31) above, and by using (33), (34), and (35), the integrations required can be carried out for all terms except the first—that containing p^{-2}. The result so arrived at is

$$T_2 = -\frac{q\kappa_1 e^{-\frac{\rho^2}{4\kappa_1 t}}}{8K_1(\pi\kappa_1 t)^{3/2}}\left[\phi_0 + \frac{x'^2}{2!} - \frac{x'^4}{3.4.(4\kappa_1 t)} + \frac{x'^6}{5.6.2!(4\kappa_1 t)^2} - \text{etc.}\right]. \quad (43)$$

$$= -\frac{q\kappa_1 e^{-\frac{\rho^2}{4\kappa_1 t}}}{8K_1(\pi\kappa_1 t)^{3/2}}\left[\phi_0 + \int_0^{x'} dx'\int_0^{x'} dx' e^{-\frac{x'^2}{4\kappa_1 t}}\right]. \quad (43')$$

The value of ϕ_0, representing the p^{-2} term, can now be found from the condition that T_2 must satisfy the original differential equation (1). Thus we obtain

$$\phi_0 = 2\kappa_1 t. \quad \ldots \ldots \quad (44)$$

From the third term in (40) we obtain a term T_3 defined by (42) above modified by replacing p^{-2} by p^{-4}. By the same process employed above to determine T_2 we find

$$T_3 = \frac{q\kappa_1 e^{-\frac{p^2}{4\kappa_1 t}}}{8K_1(\pi\kappa_1 t)^{3/2}}\left[\frac{4}{3}\kappa_1^2 t^2 + \kappa_1 t x'^2 + \frac{x'^4}{4!} - \frac{2'x'^6}{6!(4\kappa_1 t)} + \text{etc.}\right] \quad \ldots \quad (45)$$

The terms containing x' are obtained by two integrations of T_2 with respect to x', and the first term is determined by the condition that T_3 must satisfy (1). We may now express the reflected wave-system corresponding to an instantaneous source of strength q at the origin in the form

$$v_1' = \frac{K_1 - K_2}{K_1 + K_2}T_1 - \frac{K_1K_2}{(K_1+K_2)^2}d^2T_2 + \frac{K_1K_2(K_1+3K_2)}{4(K_1+K_2)^2}d^4T_3 + \text{etc.}, \quad . \quad (46)$$

where T_1 represents an instantaneous source of strength q at the image point with reference to the origin and d^2 may be replaced by the operator $\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_2}\right)\frac{d}{dt}$ throughout.

The above process of evaluating the various solutions T_2, T_3, T_4, etc. incidentally provides the values of the group of integrals

$$\int_0^\infty dp\,\frac{K_0\{\rho\sqrt{p^2+h_1^2}\}}{p^{2n}}=\frac{\Gamma(\frac{1}{2}-n)}{2^{n+\frac{1}{2}}h_1^{n-\frac{1}{2}}}K_{n-\frac{1}{2}}(h_1\rho). \quad . \quad (47)$$

The corresponding evaluations when n is negative are given in Watson's 'Bessel Functions,' p. 417.

The integral representing the transmitted wave-system may obviously be treated by similar methods, involving similar solutions in medium II. to those above discussed—but the above process leads, however, to a great complexity of terms. The study and arrangement of this integral in terms of known functions we shall leave for the present, especially as it should be possible to derive expressions for the effect in medium II. when the incident and reflected waves have been fully determined, by various indirect methods. It may be derived, for example, by an application of Green's theorem or by a method of successive approximations, or by taking the plane boundary as a limiting case of a spherical boundary. One result obtainable from the above solution (46) for the reflected wave, corresponding to an instantaneous source at the origin, and the refracted wave for the same case obtained by means of (41) above is worthy of notice. Taking (46), the reflected wave, for example; the second and all later terms being of the form $\frac{d}{dt}T_2$, when we replace t by $(t-t')$ in each, and integrate with respect to t' from $t'=0$ to $t'=t$, thereafter making t approach to infinity, we obtain in the limit when $t\to\infty$ a zero value for each term. A like result holds in the case of v_2. The solution for the case of continuous source at the origin in medium I. is therefore obtained by applying the above process of integration with respect to t' to the term T_1, as representing the incident wave issuing from the origin, to $\frac{K_1-K_2}{K_1+K_2}T_1$, as ultimately the only term representing the reflected

wave issuing from the image point, and to the initial term containing $\frac{2K_1}{K_1+K_2}$ in the expression for the refracted wave. For a continuous source q as incident wave-system in medium I. our solution is therefore

$$v_1+v_1'=\frac{q}{4\pi K_1 r}+\frac{q}{4\pi K_1 r'}\cdot\frac{K_1-K_2}{K_1+K_2} \text{ in medium I.,} \quad (48)$$

$$v_2=\frac{q}{4\pi K_2 r}\cdot\frac{2K_2}{K_1+K_2} \text{ in medium II.} \quad . \quad . \quad . \quad (49)$$

It is easy to recognize that a corresponding result is obtainable in the case of line sources, the integration with respect to k and t' being applied to (17), (18), (19) instead of to (28), (29), (30).

In the above treatment of the integral (29), representing the reflected wave, the evaluations given by (31) and (32) apply in all cases. In the integration with respect to k which follows in (34) and (35), to give the reflected wave-system when the original source is instantaneous several cases fall to be considered corresponding to the values chosen for h_1. The integral dealt with in (34) is, in fact, a particular case of the well-known Fourier integral representing an instantaneous source in any medium, which is discussed by Lord Kelvin in his paper * "On the Waves produced by a Single Impulse in Water of any Depth, or in a Dispersive Medium." The evaluations given in (34) and (35) are for the case of heat conduction, for which $h_1=\sqrt{\frac{ik}{\kappa_1}}$. Certain wave-motion subjects introduce values of h_1 given by $\frac{k}{V}$ or $\frac{ik}{V}$. The former value applies to wave-motion in a non-dispersive medium, and it is easy to verify that in this case we obtain results analogous to (46) and to (48) and (49) above. The equations (48) and (49), for example, with $\frac{q}{4\pi K_1}$ interpreted as an electrostatic pole, e, and K_1, K_2 as specific induction coefficients for the two media in contact, give the solution of the problem of determining the potential due to a pole e placed in medium I. of specific inductive capacity K_1 at distance a from the

* Phil. Mag., March 1887. See also Green, Proc. R. S. E. xxix. p. 448.

plane boundary of medium II. of specific inductive capacity K_2. This is in accordance with the well-known analogy between problems in continuous heat-flow and problems in electrostatics. Similar analogous results hold relating to magnetic poles etc; but the results referred to are in the method of the present paper obtainable by applying the integration processes described above to corresponding analogous results referring to periodic sources, all virtually represented by the solution contained in (28), (29), (30) above. The distinctions between the two cases $\kappa_1 \gtrless \kappa_2$, where κ_1 refers to the medium containing the original source, can only be obtained by a more detailed examination of the general solution obtained above.

LXIV. *The small-scale Structure of Surfaces. By* G. P. Thomson, *M.A., F.R.S., Professor of Physics at the Imperial College of Science and Technology* *.

[Plate XI.]

Introduction.

A LARGE number of experiments have been made in recent years in which electrons have been reflected from solid and, occasionally, from liquid surfaces in such a way as to form diffraction patterns due to the atoms in or near the surface. These patterns are usually sharply defined if the surface is composed of crystals each containing a large number of lattice cells, and have been used to investigate the crystallographic and chemical nature of the matter near the surface, which, owing to the small penetrating power of the electrons, is alone effective. In addition, conclusions can often be drawn as to the small-scale geometric structure of the surface. It is the purpose of the present paper to collect together some of the results on this last point, and also to examine in rather greater detail than has been done so far the conclusions to which electron diffraction experiments can lead as to the physical form of surfaces.

* Communicated by the Author.

For our purposes the three most important matters are the size of the crystals, their relative arrangement, and the degree of smoothness of the surface.

As regards the first we may have :—

(*a*) Single large crystals.

(*b*) A powder of crystals each containing very many unit cells, but of dimensions small compared with the region irradiated by the electrons.

(*c*) Crystals so small that they have poor resolving power regarded as diffraction gratings. The extreme case of this class is afforded by amorphous or liquid substances. A special case is the quasi-regular arrangement of some kinds of long chain organic compounds.

In class (*a*) we may distinguish between "perfect" crystals and those in which different portions of apparently the same crystal are inclined to one another at angles comparable with those of diffraction. For the electrons with which most of the experiments have been made and whose wave-length was of the order $\cdot 7 \times 10^{-9}$ cm. (30,600 volts), these angles are often as small as 2°, so that the standard of accuracy required is fairly high.

In class (*b*) the crystals may be arranged at random, or may show orientation in preferred directions, determined usually either by the general direction, of the surface of which they form part, or by some mechanical working which the specimen has undergone. In class (*c*) the arrangement is usually a random one.

The smoothness of the surface comes into the picture chiefly because of the refraction of the rays. Owing to the difference of potential between the inside and outside of a piece of solid or liquid matter, electrons passing in or out are bent like rays of light refracted by a surface of refractive index $1+\Phi/P$, where Φ is the difference of potential between inside and out ("inner potential") and P*e* is the energy of the electron.

Assuming that the angles of incidence and refraction are nearly 90° (glancing angles small), a simple calculation shows that the effect of refraction is to displace, in the plane containing the normal, a ray which would have emerged from the crystal at a glancing angle η to an angle η' where $\eta-\eta'=\dfrac{\Phi}{P(\eta+\eta')}$. Thus the rays are

drawn in to the surface of the crystal and the angular shift is larger, the smaller the glancing angle. If we take as average values $\Phi=10$ volts, P=30,000 volts, then $\eta-\eta'$ is just appreciable, say $\frac{1}{500}$ or about ·8 mm. on the plate, when $\frac{\eta+\eta'}{2}=\frac{1}{12}$ radian or $4\frac{3}{4}°$.

If, then, the exposed surface is sufficiently rough for most parts to make an angle greater than 5° with the electrons (incident or diffracted), the effect of refraction can be neglected; if the surface is truly flat there will be appreciable shift of those beams which make an angle of less than 5° with the surface *. Finally, if the surface is wavy so that many of the emergent electrons come through the surface at varying angles less than 5° the pattern will become diffuse owing to different electrons being refracted to different amounts. In a pattern of spots this will draw the spots out into lines perpendicular to the shadow on the photographic plate of the edge of the specimen; in a pattern of rings the shift would usually be so great as to confuse one ring with the next and would probably quite destroy the pattern.

In addition to the patterns of spots and rings, patterns of light and dark lines are known. These were discovered by Kikuchi, and are often called after him. They occur at the intersection of the photographic plate with planes (strictly speaking, cones), which make the Bragg angle on either side of a net plane of the crystal. They are subject to refraction, but their position is independent of the direction of the incident electrons. As they are fixed in relation to the crystal, any considerable imperfection in the latter will make the lines from different portions come in different places and so smudge the effect. Thus sharp Kikuchi lines are an indication of a perfect crystal. For a similar reason they never occur from random aggregates of crystals.

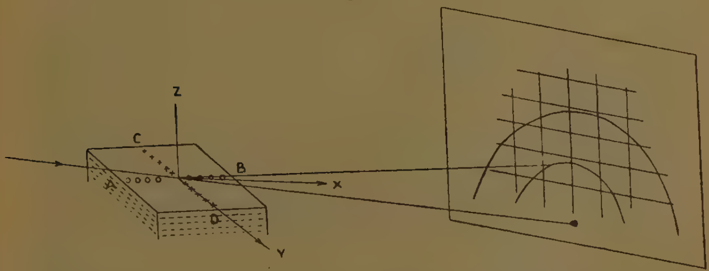
Kirchner † has expressed the conditions for the rein-

* The refraction on *entrance* will not affect the position of the beams diffracted at the Bragg angle by a crystal if these are measured from the surface of the crystal, though it will, of course, do so if the positions are measured from the direction of the incident beam. If the diffraction is by amorphous material the *deviation* in the material is fixed, but not the angle to the surface, and the refraction of both rays must be considered.

† Kirchner and Raether, *Phys. Zeit.* xxx. p. 510 (1932).

forcement of the wavelets from different atoms of a crystal in a form very convenient for the present argument. In fig. 1 the electron beam is diffracted by a crystal whose atoms can be regarded as arranged in planes parallel to the surface. The condition that the wavelets from any line of atoms shall be in phase is that the diffracted ray shall lie on one of a series of cones with the line of atoms as axis. For atoms in a line normal to the incident beam (crosses) the cones cut the photographic plate in (approx.) vertical lines, for the atoms marked by circles we get circles and for atoms in successive planes horizontal lines. Only if all three intersect in a point is a true Bragg spot formed. If the penetration is slight the horizontal lines become broad bands owing

Fig. 1.



to lack of resolving power, and any intersection of the vertical lines and circles inside the band will give a spot. If the dimensions in the surface are limited the vertical lines and circles also become diffuse, but it can readily be seen that for equal extensions in the two directions the circles broaden much more than the lines. The path difference for wavelets from A and B is AB $\cos\phi$ where ϕ is the angle between AB and the diffracted beam. The variation of this with ϕ is AB $\sin\phi \,.\, \delta\phi$. Now ϕ is small, but the corresponding angle χ for the atoms in the line CD is nearly $\pi/2$, hence for a given variation of path $\delta\phi$ is larger than $\delta\chi$ if AB$\sim$CD*. It follows that the

* Kirchner and Raether's paper reads as though they supposed that the number of atoms in AB was less than that in CD, an assumption which, at best, could only be true for one azimuth of the crystal. Kikuchi rightly objects to this, but it is not a necessary part of the argument.

angular half width, measured to a place where interference has reduced the amplitude to a given fraction of its maximum value, is greater for the circles than for the lines.

We are now in a position to consider the different classes of surfaces which can be distinguished by electron diffraction and see which have been so far observed.

Single Crystals.

A large amount of work has been done on the cleavage faces of crystals. In most cases these surfaces are flat * in our sense. They give patterns of spots which lie approximately on a circle through the spot corresponding to the incident beam, but which are displaced by refraction in accordance with the above theory. They also show Kikuchi lines, from which the angular perfection of the crystal can be estimated. In many cases this is exact to about 10′ or less, though probably most crystals have small regions of imperfection where the divergence is much more. These sometimes cause faint extra spots. The main spots are visible over a considerable range of angle, showing that the Bragg condition is not rigidly fulfilled; this implies that the electrons only penetrate a very few layers of the crystal, being stopped by a process analogous to "extinction" in X-rays. With diamond the spot reflected from a natural (111) face can be seen at all angles, though, of course, it is much brighter when the Bragg condition is satisfied. Pl. XI. shows a spot half-way between two Kikuchi lines, which implies that the waves from successive atom layers differ in phase by 180°. The faint outer spot (about 8 mm. to the left of the first) is one of the extra spots mentioned above; the sharpness of the Kikuchi lines is remarkable.

It has often been suggested that crystals are broken up into relatively small blocks which may, perhaps, be tilted slightly with respect to each other, and it is interesting to consider what evidence electron diffraction

* Shallow steps or ridges bounded by cleavage planes will not affect the form of the pattern, though they may affect its intensity, provided always that the flat top of such a ridge is large compared with the effective range of the electron, which may be taken as a small multiple of 10^{-3} cm. No electrons can then go *through* a ridge and so escape refraction at the main surface.

can afford on this point. Consider first the case of a smooth surface divided into blocks by cracks, the blocks being all supposed parallel, but the wavelets in the atoms in any one block bearing no fixed phase relation with those from any other.

In such a case the circles in fig. 1 will become diffuse, and if the horizontal lines are already so, owing to extinction, there will be regions where both overlap and are crossed by the vertical lines.

The spots observed with single crystals show the position of these regions, and the existence of several spots is due, as a rule, to a slight diffuseness of at least one of these sets of loci. If, however, they become very diffuse we should expect to see the spots drawn out in a direction normal to the crystal surface, and other spots might appear.

Both these effects have, in fact, been observed with ionic crystals, though they are not universal. A common pattern shows elongated spots grouped in concentric circles; in this case the penetration is so slight that the depth interference is unimportant, and the pattern is practically a surface one with limited resolving power in the direction AB. For such crystals a spot shows at the angle of specular reflexion for all angles of incidence even if the Bragg condition is not satisfied. The elongation of the spots varies greatly for different crystals of the same substance and from substance to substance. It is not found for diamond, and even an imperfect crystal like rock salt does not always show the effect; the better the cleavage face, the less likely is it to appear.

For crystals which give spots showing less than 1 mm. elongation with a camera length of 40 cm. the coherent regions on the surface must be at least of the order 2×10^{-5} cm., and we conclude that some crystals reach this standard of perfection, though for many the regions are much smaller.

Kikuchi * has suggested that the main cause of the elongated spots and extended patterns lies in the crystal being composed of portions tilted at small angles with respect to one another. Assuming, as before, that the depth of penetration is slight, a crystal composed of relatively large portions tilted at various angles would

* Kikuchi and Nakagawa, 'Scientific Papers of the Institute of Physical and Chemical Research,' Hongo, vol. xxi. p. 80.

behave for many purposes in much the same way as one composed of smaller pieces which had preserved their parallelism. Both would give spots elongated normal to the general surface, and in extreme cases would give extended patterns showing displacement by refraction. It seems, however, that the Kikuchi lines give a means of discriminating. Since these lines make fixed angles with the crystal planes, a crystal composed of a mosaic of parts inclined to one another cannot give sharp Kikuchi lines, but these are sometimes observed together with elongated spots—indeed, some of Kikuchi's own illustrations show them. In such cases one must attribute the elongation to a splitting of the surface into small blocks without loss of orientation. In other cases, where the lines do not appear, it is probable that the blocks are tilted, and this may be the important factor *.

Etched Single Crystal.

When a single crystal, either of a metal or of a salt, is etched it gives an extended pattern of spots which agree with what would be given by a cross-grating of diffracting points arranged like the atoms in the net plane of the crystal normal, or nearly normal, to the incident beam of electrons. It can be shown † that this is what one would expect if the electrons passed through projections on the surface of thickness not exceeding a small multiple of 10^{-7} cm.; the extent of the pattern is due to the poor resolving power of lines of atoms nearly parallel to the beam, as explained above (p. 643). An alternative explanation in the metal case has been given by Germer ‡, who regards it as due to the distortion of the lattice, so that parts of the crystal are orientated at the right angle to give each spot by Bragg reflexion. The surface is certainly rough since there is no refractive effect; and since the effective penetration of the electrons is not much more than 10^{-6} for heavy metals, the first explanation must at least account for a good deal of the effect unless one were to

* The question of the width of the Kikuchi lines has not yet been satisfactorily explained. In some cases they appear to be much sharper than would be expected in view of the heavy extinction; it is accordingly difficult to say how much their sharpness would be affected by a division of the crystal into small *parallel* blocks.

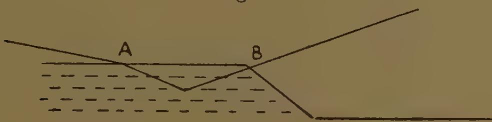
† Proc. Roy. Soc. vol. cxxxiii. p. 1 (1931).

‡ Germer, Phys. Rev. vol. xliv. p. 1012 (1933).

suppose that the available projections were all of a uniform thickness near 10^{-6}. In the case of salts, Tillman* has recently found cases in which the cross-grating pattern is accompanied by Kikuchi lines, implying that the crystal is sensibly perfect as regards direction, and in such a case the second explanation is excluded †. Metals do not, as a rule, show lines, and with their soft deformable lattices distortion is very probable, especially as they have to be cut and ground before etching to prepare a flat surface (unlike ionic crystals, which can often be cleaved), so that both causes are probably acting.

The "etched" pattern can be distinguished from the extended pattern described in the previous paragraph by the absence of any refraction. It is formed by electrons going through projections, pyramids or ridges, on the surface, while in the former case the surface is flat and electrons enter and leave through the same crystal face.

Fig. 2.



Elongation of Spots by Refraction.

The elongation due to these kinds of imperfection in the crystal is symmetrical about the mean position of the spot, which in this case is that corresponding to refraction. Another form of elongation is sometimes found in which the spot extends inwards unsymmetrically from the unrefracted position. In this case we have to deal with a surface partly covered with projections, giving the "etched" *unrefracted* pattern, but partly flat or nearly so. A flat portion would, of course, give a spot in the *refracted* position if the incidence is chosen so as to satisfy the Bragg relation. Flat pieces at other angles would, as before, give rise to a blackening of the plate on each side of the refracted spot, and intermediate effects are possible from electrons entering at one face and leaving at another, *e. g.* the ray in fig. 2 would be

* See *infra*, p. 656.

† Kikuchi and Nakagowa publish a similar picture, though it seems inconsistent with their explanation that the effect is due to imperfection of the crystal.

appreciably refracted at A, where the glancing angle is small, but not at B, where it is large.

Elongation of this kind has been observed by French * on a partially polished crystal of chromium, and by Darbyshire † on a cleavage face of antimony. It must be remembered that, owing to the small glancing angles used, quite a low ridge will shield a large area of flat surface from the rays, so that it is not surprising that "etched" patterns due to such ridges should sometimes be fairly strong, even with cleavage faces which are optically almost perfect. This is especially marked in the case of bismuth, where they are the only patterns found.

Polycrystalline Surfaces.

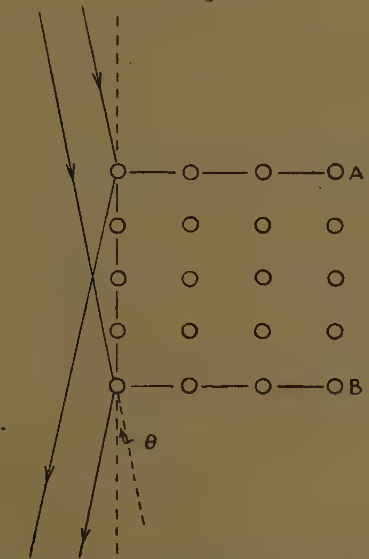
The commonest type of pattern found by electron diffraction consists of a series of concentric circles or rather arcs, since more than half of each circle is cut off by the shadow of the specimen. The rings are often very sharp, their width being due only to the finite size of the incident beam and to its slight lack of homogeneity in velocity. These rings are usually explained as the Bragg reflexions from the planes of crystals arranged at random like the Debye-Scherrer rings from X-rays. But it is better to regard them as due to diffraction by planes of atoms forming two-dimensional gratings normal to the incident beam ‡. Consider a crystal like that shown in fig. 3, where reflexion can take place from planes parallel to AB. The beam reflected from any one crystal plane will spread by diffraction to a half-width $\phi=\frac{\lambda}{AB\sin\theta}$ if $\phi<<\theta$. Hence to get the reflexion sharp to say $10'$, and many of the rings have a half-width less than this, $AB\,.\,\theta\sim 2{\cdot}4\times 10^{-7}$ or since $\theta\sim\frac{1}{40}$ we find $AB\sim 10^{-5}$ cm.

Now we can calculate roughly the distance which an electron can penetrate a solid without making an inelastic collision, and so becoming incapable of interfering. Smith § finds that the number of primary ions made by an electron of 4500 volts energy (the fastest he used) in argon at 1 mm. pressure is about ·9 pairs

* French, Proc. Roy. Soc. vol. cxl. p. 637 (1933).
† Darbyshire, Phil. Mag. vol. xvi. p. 761 (1933).
‡ 'Wave Mechanics of Free Electron,' p. 72.
§ Smith, Phys. Rev. vol. xxxvi. p. 1293 (1930).

per cm. For fast electrons the chance of ionization is roughly inversely as the energy, and for 30,000 volts would be ·13 per cm. Now there are $3{\cdot}55 \times 10^{16}$ atoms per c.c. in the gas, while in an ordinary solid of about the same atomic number as argon there are about $\cdot 6 \times 10^{23}$ atoms per c.c. (*e. g.*, iron has $\cdot 84 \times 10^{23}$, copper $\cdot 85 \times 10^{23}$, aluminium $\cdot 6 \times 10^{23}$, NaCl $\cdot 45 \times 10^{23}$). Hence the mean free path for ionization in a solid is $\dfrac{3{\cdot}55 \times 10^{16}}{\cdot 13 \times \cdot 6 \times 10^{23}}$ or 4×10^{-6} cm. approximately.

Fig. 3.



This is in agreement with the general experience that films of heavy metals more than about 10^{-6} cm. thick give patterns with rather heavy backgrounds, and also with the experiments of Kirchner *, who finds

* Kirchner, *Ann. der Phys.* vol. xiii. p. 38 (1932). In Kirchner's experiments the absorption coefficient was found from the loss in intensity in the main beam after passing through films of mica. With the thickness used ($\sim 10^{-5}$ cm.) the main beam only retained a few per cent. of its original strength, and though Kirchner showed that the elastically scattered beams also had intensities of this order of magnitude, it seems probable that the main loss of intensity from the main beam was caused by the inelastic scattering.

that μ^{-1} for mica is about $2{\cdot}8 \times 10^{-6}$ for electrons of 30,000 volts energy.

It follows that crystals as thick as 10^{-5} or more would destroy the capacity for interference of most of the electrons passing through them, and we must attribute most of the intensity of the rings to thinner crystals acting mainly as two-dimensional gratings. There is another reason why thin crystals, of the order of 10^{-6}, will give more intense diffraction patterns than thicker ones. For the thick crystals reflexion will only occur over a narrow range of angle of incidence ($\sim 10'$), and in a random assembly few will be at the correct angle. But for a two-dimensional grating at nearly normal incidence the deviation is proportional to the cosine of the angle of incidence, and so will be constant to 1 per cent. over a range of $\pm 8°$. It is true that such a grating will never give the 100 per cent. reflexion which can theoretically be given by a large crystal at the Bragg angle, but, as Bragg has pointed out, a single layer of atoms will diffract several per cent. of the rays, and one 10^{-6} cm. thick would probably diffract a very large fraction. Hence the much greater of range of angle for the thin crystal will far more than compensate for the better diffracting power of the thick crystal when set at the most favourable angle.

If the crystals are large another possibility arises, namely, reflexion from an exposed face, the rays entering and leaving from the same face. In this case the rays would be refracted and the rings would be smaller than the corresponding ones formed in the ways we have considered so far. Since the penetration would probably be much limited by extinction the reflexion would not be sharply selective, and the rings would be broadened as well as reduced in size. As far as I am aware, this effect has never been reported, but it is possible that it may account for some of the faint inner rings which have been found sporadically by several experimenters and attributed to "forbidden" or half-order reflexions.

Patterns of sharp rings are given by many types of surface :—Ordinary metals, machined, sand-papered or etched ; metals which have been oxidized or otherwise treated chemically ; surfaces on which metal has been deposited by sputtering or evaporation ; surfaces covered with fine powders ; fused quartz ; besides, of course, those formed by transmission through thin films of

metals or of powdered salts. In many cases heat treatment of the surface results in orientation of the crystals, some particular crystal plane tending to align itself parallel to the general surface. This change is probably due rather to recrystallization than to a bodily turning of the crystals. It makes the intensity of the rings uneven, and in extreme cases the rings are replaced each by one or more short arcs; it can also alter the relative intensity of the different rings.

We may conclude that surfaces in the above list have sharp projections or ridges of the order of thickness 10^{-6} cm., at least in their thinner parts. In order to give the observed sharpness of pattern the thin portion must have an extension of at least 10^{-6} each way in the plane normal to the rays. Most surfaces deteriorate on keeping from the point of view of diffraction patterns, which is probably due to the sharp edges wearing off or being covered by gummy material from the air of the laboratory. It should be noticed that it is not necessary for more than a small fraction of the surface to be covered by projections, for owing to the small angle between the rays and the surface, a projection will shield many times its own area from the rays.

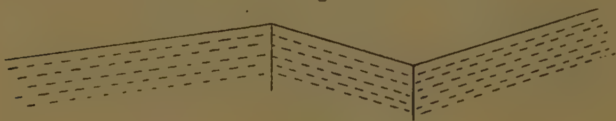
In some case surfaces prepared in one of the above ways show diffuse rings. These can sometimes be shown to be due to the blurring of a normal ring pattern, which can be formed with sharp rings by some change in the conditions of preparation. It is natural to suppose that the blurring is due to deficient resolving power of the crystal gratings such as would occur if each individual crystal only contained a few unit cells in any direction. In the case of transmission patterns formed by evaporating salts on to thin celluloid, Kirchner * has shown that the pattern becomes much sharper in the course of a few hours, from which it may be inferred that the crystals are growing larger.

If a surface which gives sharp rings is rubbed or polished. the fine projections which account for the pattern will, of course, be destroyed, and if the tool is a smooth one, few fresh ones will be formed to take their place. Experiment shows that in the case of metals the rings gradually become diffuse until finally the pattern is reduced to two or three very broad rings. It is natural to regard

* Kirchner, *Zeit. für Phys.* vol. lxxvi. p. 576 (1932).

the increasing diffuseness of the rings as due to decreasing size in the portions of crystal which are in regular order and project above the surface, and the final pattern as due to the amorphous layer which Bielby long ago found on polished metals. Other possibilities have, however, been suggested, and it is interesting to see how far they can be excluded. One natural supposition is that the surface, after rubbing, is composed of crystals inclined at various small angles, each having one of the main crystal planes parallel to its surface, in particular the cleavage plane if there is one (fig. 4). Such a surface would give strong reflexion of rays incident near the Bragg angle from any plane, not only that parallel to the surface of the particular crystal struck. The depth of penetration corresponding to a path in the metal of 10^{-5} cm. at an angle of $2°$ to the surface is only $1{\cdot}6\times10^{-7}$, and the

Fig. 4.



effect of extinction will still further diminish this; accordingly, the condition of reinforcement in depth is relaxed, and calculation shows that reflexion can occur over a wide range of angle of incidence giving diffuse spots. If there were several different kinds of crystal face exposed, each arranged at random in azimuth, the diffuse spots might form sensibly uniform diffuse rings. These rings, however, would be subject to refraction. which would both shift their position and alter their shape. In fact the rings remain circular, and though the size of a diffuse ring is not always the mean of the sizes of the sharp ones to which it appears to correspond, the difference is much less than would correspond to an inner potential of 15 volts, which is the order of its value for most metals.

Germer * has suggested that the diffuseness of the rings might be explained by supposing that the crystalline surface is wavy and that electrons are refracted to different extents dependent on the slope of the surface where they

* Germer, Phys. Rev. vol. xliii. p. 724 (1933).

enter or leave the metal*. The difficulty about the shift of the rings might be partly avoided on this view if it were supposed that the glancing angles were of the order of 5°. Then the refraction would be enough to broaden the ring but not enough to shift it much. Such shift as occurred would have to be towards a smaller diameter, and this does not, in fact, agree with observation. Further, it is difficult to see how such a wavy surface can be formed and still be crystalline. A change of angle of a few degrees in a distance of 10^{-5} cm. seems far too big a distortion of the lattice to be permissible, and the only alternative seems to be to suppose the wavy layer to be formed of small broken crystals containing a few atoms each, or even of single atoms, which is practically a return to the original hypothesis of an amorphous layer.

No doubt when the layer is very thin the electrons may penetrate to the crystals beneath, which would thus cause some of the broadening; possibly also appreciable refraction occurs at the flatter parts of the amorphous layer, and this will still further broaden the rings.

The working of the metal must have effects below the actual surface, and probably as the surface is approached the mean size of the crystals steadily diminished ; but the effects observed cannot be accounted for solely by diminished crystal size, as besides the blurring of the rings there are definite changes in their size. We must suppose that there is a truly amorphous layer on the surface, which is sufficiently wavy to make the refraction effects small (angles of order of 5° or more). The recent work of Finch †, Quarrell, and Roebuck on the growth of crystals on polished surfaces, confirms the view that a polished surface differs fundamentally from a crystalline one.

Conditions for Diffraction.

It may be of interest to consider the circumstances under which a surface may fail to give a pattern at all. This will happen in the first place if the specimen becomes negatively charged and repels the rays. We can often

* This view differs from the preceding in that the reflexion *inside* the crystal is supposed to take place exactly at the Bragg angle and the diffuseness is due to refraction.

† Finch, Quarrell, and Roebuck, Proc. Roy. Soc. vol. cxlv. p. 676 (1934).

overcome this difficulty by care in mounting the specimen, as if the rays only strike the surface at a small angle the secondary electrons given off are more numerous than the primary and the surface acquires a positive charge. Quite a small positive potential is enough to prevent the escape of most of the secondary electrons, and the deflexion of the ray becomes negligible.

If the surface is non-crystalline we may still get a diffuse pattern like that found from polished metals, depending on the existence of certain regularities in the distances between the atoms—for example, when atoms approximating to hard spheres are piled together, even without order, a distance equal to the atomic diameter will frequently occur. Similar regularities occur within the molecules of many organic compounds, notably the distance between successive atoms in a carbon chain, and may cause quite sharp patterns *, especially if the molecules are arranged in some approach to order. Some substances, however, have no dimension which predominates sufficiently to give detectable effects, and the electrons scattered from them give a continuous blackening on the photographic plate. For example, it has not been found possible to get a measureable pattern from films of gelatine.

In other cases, probably more numerous, the absence of effect is to be attributed to the background of diffusely scattered electrons swamping the effect of the diffracting crystals. There is a great difference between the distance ($\sim 10^{-5}$ cm.) which an electron can go through matter without making an inelastic collision, and the distance it can go before it loses so much energy that it can no longer affect a photographic plate. For 30,000 volts electrons in a average solid this last distance can be taken as of the order 10^{-3} cm. All excrescences on the surface of thickness between these values will give rise to inelastically scattered electrons which will form a background that may be strong enough to obliterate the pattern. Lumps larger than about 10^{-3} cm. will reduce the intensity of scattered electrons by absorption without much altering the proportion between elastic and inelastic scattering †. It must be noticed also that if

* Murison, Phil. Mag. vol. xvii. p. 201 (1934).

† It will, no doubt, often happen that an excrescence which is "thick" at the base will be thin at the top, and this part may then diffract.

electrons diffracted by one portion of a surface hit another portion they will be diffracted again and the pattern smudged, even though each portion of the surface taken separately would give a sharp pattern. A layer of gas on the surface would probably produce additional background; no patterns specifically due to adsorbed gas have been observed with fast electrons.

We have seen that amorphous and microcrystalline substances give diffuse patterns. If it happens that the surface of such a substance is wavy to the extent of a few degrees, but not more, there will be an additional blurring of the rings due to variable refraction, and this may well be enough to obliterate them entirely. Finch, Quarrell, and Roebuck (*loc. cit.*) report that very highly polished metal surfaces show no pattern—perhaps this is a case in point.

Conclusions and Summary.

Evidence derived from experiments on electron diffraction leads to the following conclusions. In some cases the material near the cleaved or natural face of a single crystal is correctly aligned to $10'$ or less. The surface, apart from steps, is sometimes smooth but sometimes carries narrow ridges $<10^{-6}$ cm. thick (if, as is probable, the ridges taper, their bases may be much thicker than this). In many cases, but not always, the surface is broken into small regions of the order of 10^{-5} cm. These regions sometimes keep their parallel alignment, but probably not always. They are most likely to be distorted in the case of metals. Etching, as might be expected, leaves a surface of projections and ridges, many of which are not more than a small multiple of 10^{-7} cm. thick. The degree of perfection of a crystal surface varies considerably from specimen to specimen and with the care with which it has been prepared.

A number of surfaces composed of polycrystalline aggregates carry projections not more than about 10^{-6} cm. thick covering an appreciable fraction of their surface. Such surfaces are formed when crystals are deposited by evaporation or cathodic sputtering, and when chemical action, either in the form of etching or the formation of compounds, takes place at metallic surfaces. In some cases the crystals are extremely minute, only containing a few unit cells in any direction. Heat

treatment, and sometimes merely keeping, may produce recrystallization associated with increase in the size of individual crystals and often their orientation in preferred directions. Polishing of metals covers the surface with an amorphous layer ; this layer is not usually flat, being formed into ridges or waves over at least part of the surface. Sometimes, perhaps, the inclination to the mean surface of these waves may be as little as two or three degrees, but it is usually a good deal more. Surface layers of organic substances are often strongly orientated.

South Kensington.
July 30.

LXV. *The Measurement of Mean Inner Potential. By* J. R. Tillman, *B.Sc., A.R.C.S., Imperial College of Science and Technology.*

[Plates XII. & XIII.]

IT has been recognized since the earliest experiments on electron diffraction that the de Broglie wave-length of an electron might change with the medium. In 1928 Davisson and Germer † definitely showed the existence of a refractive index in their work on nickel. Since then others (who will be referred to separately later on) have made measurements on this refraction effect. The refractive index, however, is not considered to be the fundamental quantity. The theorists have demanded the existence of a quantity Φ, called the mean inner potential and expressed in volts, which should be constant for any one material and which is intimately bound up with the refractive index μ. Bethe ‡ has shown the relation between μ, Φ, and the voltage P, through which the electron being refracted has fallen, to be

$$\mu=\sqrt{1+\Phi/P}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Measurements of Φ have shown fairly conclusively that for any given value of P, Snell's law is obeyed : this

* Communicated by Prof. G. P. Thomson.
† Proc. Nat. Acad. xiv. pp. 317 and 619 (1928).
‡ *Naturwissenschaften*, xv. p. 787 (1927) and xvi. p. 333 (1927).

indicates the presence of a true refractive index. From the relation (1) Φ has been calculated. But here a discrepancy has arisen: although Φ for any one substance has been found independent of P, when P was varied from 20–50 kilovolts, and similarly when P was varied from 20–400 volts *, the two values of Φ obtained over the two different ranges of P did not agree. The low voltage experiments have since been partially explained away †. The high voltage measurements have, on the whole, been consistent amongst themselves. The present work has been an attempt, both to obtain some accurate values of Φ for various substances and also to show the independence of Φ of P. Two separate ranges of P have been used. The first, 20–40 kv., has been the same as that already used by other workers; the second, 3–6 kv., has not been used before in electron diffraction work.

Theory.

If a beam of electrons makes an angle θ' with the surface of the specimen and after refraction makes an angle θ then

$$\mu=\sin\left(\frac{\pi}{2}-\theta'\right)\Big/\sin\left(\frac{\pi}{2}-\theta\right),$$

and hence

$$\mu^2-1=(\cos^2\theta'-\cos^2\theta)/(1-\sin^2\theta).$$

From equation (1) we have

$$\Phi/\mathrm{P}=(\sin^2\theta-\sin^2\theta')/(1-\sin^2\theta). \quad . \quad . \quad . \quad (2)$$

If θ' corresponds to a diffraction maximum

$$2d\sin\theta=n\lambda,$$

and

$$\sin^2\theta=n^2\lambda^2/4d^2=150n^2/4d^2\mathrm{P}\left(1+\frac{e\mathrm{P}}{600m_0c^2}\right),$$

where λ, n, m_0, c, and e have their usual meanings. Putting $\mathrm{P}\left(1+\frac{\mathrm{P}}{10^6}\right)=\mathrm{P}_r$ we have, using equation (2),

$$\mathrm{P}_r\sin^2\theta'=150n^2/4d^2-\Phi\mathrm{P}_r(1-\sin^2\theta)/\mathrm{P}, \quad . \quad . \quad (3)$$

where P_r and Φ are in volts.

* Farnsworth actually found a variation of Φ for copper. Phys. Rev. xxxiv. p. 678 (1929).

† Kirchner, *Ergebnisse der Exakten Naturwissenschaften*, xi. p. 85 (1932).

Since $\sin^2\theta$ was never appreciable, the term $(1-\sin^2\theta)$ has been taken as unity. Then if we put $P_r\Phi/P=\Phi'$ and $150/4d^2=K$, equation (3) reduces to

$$P_r \sin^2\theta' = n^2K - \Phi'. \quad . \quad . \quad . \quad (4)$$

When $P_r \sin^2\theta'$ was plotted against n^2, a straight line was obtained in all cases. The intercept on the $P_r \sin^2\theta'$ axis gave the value of Φ'. The slope of the line gave a check on the spacing d, and assured that correct order numbers had been assigned. In many cases the observations on different orders were weighted differently, and to obtain the best values of K and Φ from the measurements, the method recently advocated by Awberry * was used. This proved much quicker than the method of least squares, and gave, to the accuracy required, the same values of K, Φ, and the probable error of Φ. The values of $\sin\theta'$ were obtained by screen measurements taken in a manner similar to that described by Prof. Thomson †, and also from plates taken.

Shinohara ‡ showed that the mean inner potential of a substance might be measured by observing the refraction effect of P-lines, especially those due to planes parallel to the cleavage face. In the present work this method of measurement has been used. The equation (4) applies here. The value of $\sin\theta'$ depends on the position on the plate of the P-line under consideration relative to the line of intersection of the cleavage face with the plate, often called the shadow edge. Shinohara determined the latter position by observations on those parts of oblique P-lines only very little affected by refraction. When, however, a diffracted spot on the equatorial line (that line passing through the undeviated spot perpendicular to the shadow edge) is obtained in its position of maximum intensity, the shadow edge is midway between the undeviated and diffracted spots, whether the diffracted spot shows the effect of inner potential or not, providing always that the cleavage face is flat. This method of determining accurately the position of the shadow edge has been used throughout, and the results have justified its use.

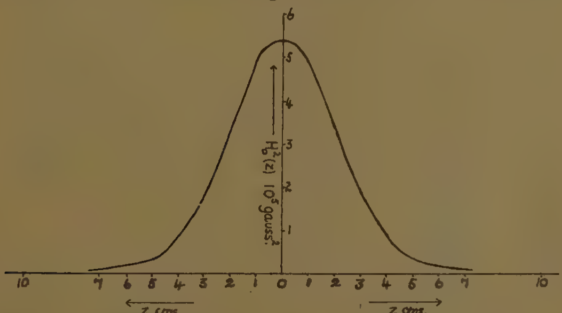
Thus three methods were available for the measurements. The same specimen could be, and was in many cases, examined by the separate methods in turn.

* Proc. Phys. Soc. xli. p. 384 (1929).
† Proc. Roy. Soc. A, cxxxiii. p. 1 (1931).
‡ Sc. Pap. I. P. C. R. xviii. p. 315 (1932).

Experimental.

A modified form of the electron camera designed by Prof. Thomson * was used. The camera length was shortened to 18 cm. instead of the usual 40 cm., to allow for the larger angles to be used with the slower electrons. The usual two anode slits defining the beam were found to provide much too little intensity in the undeviated beam, as viewed on the fluorescent screen, in the range of P, 3–6 kv. Only one anode hole, 0·03 cm. in diameter, was used, and the divergent beam of electrons issuing from it was brought to a focus on the screen by means of a magnetic lens placed between the anode and the specimen. The current carrying coil producing the magnetic field consisted of 490 turns of no. 22 d.c.c.

Fig. 1.



copper wire spread over 10 layers of length 5·25 cm. and of radius varying from 2·87 to 3·70 cm. Busch † showed theoretically in two important papers that the focal length of a magnetic lens could be expressed as a function of the field along the axis of the coil. His expression may be put in the form

$$f=\frac{8P}{i^2\eta\int H_0^2(z)dz} \quad \text{where} \quad \eta=\frac{e}{m},$$

$H_0(z)$ is the value of the field on the axis of the coil at a point z (the origin $z=0$ being at the centre of the coil) when unit current is flowing, and i is the current. This integral has been evaluated graphically (fig. 1), the field

* Proc. Roy. Soc. A, cxxviii. p. 641 (1930).

† *Ann. d. Phys.* 1926, p. 974, and *Arch. f. Electrotechnik*, 1927, p. 583.

at any point being calculated from the dimensions of the coil. With this done, the focal length could be calculated for different values of i and P, and compared with that found experimentally by measuring the distance of the anode from the coil, a, and the distance of the screen from the coil, b, when the focussed spot was at its sharpest. Then as in optics $1/f=1/a+1/b$ all quantities being positive.

A final calculation showed that $f=0\cdot001515\ \mathrm{P}/i^2$ cm., where P is in volts and i in amperes.

Table I. shows the comparison between the focal lengths calculated from this formula with the values from the lens formula.

The discrepancy between the two sets of values of f, which is fairly consistent, is probably due to the fact that

TABLE I.

i amps.	P volts.	$f=0\cdot001515\frac{\mathrm{P}}{i^2}$.	a.	b.	$f=\frac{ab}{(a+b)}$.
		cm.	cm.	cm.	cm.
·55	3500	17·7	42·6	33·3	18·7
·61	3550	14·4	22·9	53·0	16·0
·74	3580	10·0	13·7	62·2	11·2
·86	4210	8·6	14·6	33·1	10·0
2·42	31700	8·2			
2·30	27900	8·0	13·7	34·3	9·8
2·10	24000	8·3			
1·90	18700	7·9			

Busch's formula is an approximation. Reference to his papers will show that the formula is accurate only when a certain oscillating series is rapidly convergent. In the present case, one more term of the series being taken into account would make the integral smaller and the difference between the two sets of values for f would be lessened.

The discharge-tube and the hot filament were used successively as sources of electrons. An oxide coated nickel ribbon served as the latter; it was surrounded by a hollow steel cylinder to which a small negative potential bias could be applied. This bias increased the number of electrons emitted in a direction along the axis of the cylinder, the filament and the anode hole being on this axis. Moreover, the cylinder ensured a longer life to the filament, as it absorbed some of the positive ions which

normally would have bombarded the nickel ribbon. The hot filament source of electrons allowed much better vacuum conditions to be obtained, as a leak had to be used with the cold cathode. The leak was designed by Mr. Fraser, of the Physics workshop, and was of the needle type. The needle was not tapered and fitted very tightly inside a hollow cylinder. In addition, it was not rigidly connected to the screw head moving it and so did not depend on the latter for its alignment. A ball socket was used for this purpose. Dust-free dry air was admitted by the leak.

The diffraction patterns were viewed on the fluorescent screen, and special precautions had to be taken to prepare screens suitable for the low voltage work. Very finely powdered zinc sulphide was deposited in a thin layer on plate glass using the minimum amount of canada balsam necessary to ensure permanent adhesion. The layer was made uniform by spreading with a glass rod in the presence of a small quantity of xylol.

The specimens were clamped in a metal holder and could be given the same movements as those described in Prof. Thomson's paper. The system was evacuated by a two-stage mercury diffusion pump backed by a Hyvac. A transformer, rectifying valve, and smoothing condenser served as a high tension source for the range 20–40 kv. and with a much reduced primary current for some of the 3–6 kv. work, but most of the latter was done with an Evershed generator capable of giving 5 kv. A spark-gap between 5 cm. balls was used to measure the voltage in the higher range, and an Everett-Edgecombe electrostatic voltmeter served for the lower range.

Results.

Rock Salt.—This specimen was investigated very fully to test the possibilities of 5 kv. electrons in these measurements. The first observations were made with a cold cathode source of electrons. The results were consistent, and showed a value of Φ rather smaller than that already found by Yamaguti * and Shinohara † in the higher voltage range. The substitution of a filament source did not cause any change in the value of Φ : here, some 25

* Proc. Phys. Math. Soc. Jap. xiv. p. 1 (1932).
† Sc. Pap. I. P. C. R. xviii. p. 315 (1932).

specimens were used, both of the blue and colourless varieties. Little difference could be observed between the two kinds. The blue appeared to cleave rather better and to charge up less. With careful use, a specimen would give ten readings of one or more of the 2nd, 3rd, and 4th orders of reflexion from the (001) face, without showing any charging up or appreciable fatigue. The latter effect was investigated with a view of finding some method of recovery, but no successful way was found. The low voltage observations are given in Table II. A spacing of 2·814 Å. has been assumed for the (002) planes.

The mean value of Φ from the 3rd and 4th orders was $5{\cdot}9 \pm 0{\cdot}2$ volts. The probable error quoted throughout is

TABLE II.

Source of electrons.	Range of voltage kv.	No. of specimens.	No. of readings.	Value of Φ (volts) calculated from order number.		
				2.	3.	4.
Cold ...	3·78–4·61	14	85	..	$5{\cdot}7\pm0{\cdot}3$	$5{\cdot}5\pm0{\cdot}4$
Hot ...	2·94–4·33	25	254	$4{\cdot}7\pm0{\cdot}5$	$5{\cdot}9\pm0{\cdot}2$	$6{\cdot}3\pm0{\cdot}2$

$\sqrt{\frac{\Sigma\nu^2}{n^2}}$, where ν is a typical residual and n the number of observations under consideration. In view of Yamaguti's recent work * the low value obtained for the second order is not surprising, and it has been ignored in calculating a mean value of Φ.

Specimens were then investigated by the three methods in turn, a cold cathode being used. The results are tabulated in Table III. H refers to centre line spot measurements with 20–40 kv. electrons, L with 3–6 kv., and K to the P-line observations.

The mean values of Φ were calculated, and as obtained from the L method was less than those from the H and K methods. These are shown in Table IV., which includes the results of Table II.

These values agree well with those of Yamaguti and

* Proc. Phys. Math. Soc. Jap. xvi. p. 95 (1934).

Shinohara, who obtained $7{\cdot}7 \pm 0{\cdot}03$ * volts (H method) and 6·3 volts (K method) respectively.

Attempts were made to etch rock salt to obtain a single crystal pattern similar to that obtained by Prof. Thomson † for copper. Diffraction patterns showed that etching with water left the surface fairly smooth as opposed to the surface made of small projections. Rock salt etched

TABLE III.

Method.	Specimen number.	Voltage range kv.	No of readings	Φ (volts)	d in Å. (calc.).	Remarks.
L	3	4·75–5·45	16	$6{\cdot}7 \pm 0{\cdot}3$	2·82	Blue.
	4	4·81–5·60	14	$6{\cdot}1 \pm 0{\cdot}3$	2·84	Colourless.
H	1	24·4–34·1	36	$6{\cdot}9 \pm 0{\cdot}2$	2·80	Blue.
	4	27·4–27·6	10	$7{\cdot}0 \pm 0{\cdot}6$	2·81	Colourless.
K	1	33·2	..	7·7	2·79	Blue.
	2	32·6	..	7·3	2·79	Colourless.
	3	26·4	..	8·6	2·81	Blue.

TABLE IV.

Method.	Mean value of Φ (volts).
L	$6{\cdot}0 \pm 0{\cdot}2$
H	$7{\cdot}0 \pm 0{\cdot}2$
K	$7{\cdot}7 \pm 0{\cdot}5$

along the (011) face, however, did give a cross grating pattern providing that after grinding the surface it was not etched for too long a period. From these plates it

* It should not be inferred from Yamaguti's $\pm 0{\cdot}03$ that his individual measurements were more accurate than the present, as for one of his plates giving a mean value of 7·1 volts, a calculation shows that the separate orders gave values of 6·4, 7·7, 7·0, 9·0, 5·6, 6·1, and 12·1 volts, corresponding to a probable error of a single observation of 1·4 volts or, ignoring the last order, 0·8 volts, which is about the same as found in the present work.

† Proc. Roy. Soc. A, cxxxiii. p. 1 (1931).

was evident that the refraction effect was now negligible, and a good check on the spacing was made.

Many of the specimens of rock salt were found to be coloured a light brown after bombardment with the faster electrons, and the resultant effect on the blue variety was to make it appear green.

The kind of plate obtained with 3–6 kv. electrons from a cleavage face is shown in Pl. XII. fig. 1. This photograph is better than the average obtained and was taken with a freshly cut face of blue rock salt.

Zincblende.—The blende used was of three different origins. The first came from Spain (S.), was a very light brown in colour and probably contained but little iron. It was as pure a crystalline specimen as Ward's of Rochester, New York, could supply. The second came from Pennsylvania (P.), was of light to medium brown in colour, and undoubtedly contained some iron. The third, of unknown origin (U.), was a deep brown variety, being almost black in patches. The three methods of investigation were used. The results are tabulated in Table V.

From these results it is difficult to conclude whether or not the small iron content has any appreciable effect on the value of Φ. The mean value of Φ for the P and S specimens in the range 20–40 kv. was $12{\cdot}6 \pm 0{\cdot}15$ volts. Dixit * has obtained a value $2{\cdot}6 \pm 1$ volts using a fairly pure specimen. He stated that P-lines due to the planes parallel to the cleavage face were not obtained. In the present work, these lines have always been observed, even in unfavourable settings of the crystal. Pl. XII. fig. 2 shows four such lines. Kikuchi and Nakagawa †, who have done much work on the reflexion of electrons from zincblende cleavage faces, have obtained these lines. A calculation, based on their first photometer curve for the normal rotation picture, shows a value of Φ, for their specimen, of about 13·5 volts. Moreover, a very recent paper of Yamaguti ‡, published since the present work was completed, has shown a value of Φ of 12·7 volts. It would seem, therefore, that there is good evidence for supposing the value to be in the region of 13 volts.

A specimen was etched and its diffraction pattern examined. Pl. XII. fig. 3 shows one of the pictures obtained.

* Phil. Mag. cvii. p. 980 (1933), and cxiii. p. 732 (1934).
† Sc. Pap. I. P. C. R. 1933, p. 256.
‡ Proc. Phys. Math. Soc. Jap. xvi. p. 95 (1934).

From similar photographs and some screen measurements it was deduced that the surface was no longer plane. The photographs showed a wealth of P-lines; this is in direct contrast with the effect found by Prof. Thomson for copper and silver. The metal crystals gave only the series of spots corresponding to the cross-grating.

TABLE V.

Method.	Specimen number.	Origin.	No. of readings.	No. of orders.	Voltage range kv.	Φ (volts).	*d* in Å. (calc.).
L	1	U	45	2	4·15–4·60	11·2±0·3	1·94
	4	S	18	2	4·50–5·50	12·8±0·6	1·93
	5	P	16	2	4·95–5·39	11·3±0·7	1·92
H	3	S	50	4	23·2–34·1	13·3±0·2	1·88
	4	S	18	3	29·7–31·7	11·0±0·5	1·91
	4	S	17	3	29·4–31·7	12·3±0·6	1·88
				Mean value for Spanish variety.		12·6±0·2	
K	3	S	..	3	29·2	10·4	1·91
	3	S	..	3	27·0	12·4	1·88
	3	S	..	4	31·4	11·0	1·89
	3	S	..	4	22·5	13·6	1·90
	4	S	..	4	28·5	12·1	1·88
	4	S	..	3	29·1	12·7	1·88
				Mean value for Spanish variety.		12·1±0·3	
	5	P	..	3	30·8	12·8	1·88
	5	P	..	4	31·4	13·4	1·89
	6	P	..	4	29·2	12·1	1·91
	6	P	..	4	30·0	13·4	1·89
				Mean value of Pennsylvanian variety.		12·9±0·2	
	7	U	..	4	24·0	13·4	1·90
	7	U	..	4	24·5	13·5	1·90

X-ray work has shown a value of $1{\cdot}91_5$ for the spacing of the (022) planes.

Galena.—This specimen was investigated by the three methods and the results are given in Table VI. The spots obtained with the slower electrons were rather poor in intensity, but showed little sign of fatigue. The P-lines due to planes parallel to the surface were rather broader than those observed with zincblende and rock salt. This was due to small imperfections of the surface. None the less, six lines could be obtained, and measure-

ments on these showed good agreement with those of the H method.

In both the high and low voltage measurements of spots, an anomaly has been observed with the third order. The second order, though present, was blurred considerably by the surface lines (investigated by Kirchner and Raether *), and this blurring extended to the third order. This effect, combined with that found by Yamaguti † for low order numbers, caused the position of maximum intensity of the third order spot to be at a much larger angle than would have been expected from observations

TABLE VI.

Method.	Specimen number.	No. of orders.	No. of readings.	Voltage range kv.	Φ (volts)	d in Å. (calc.) (200).
L ▲	4	3	18	4·90– 5·50	12·7±0·5	2·98 assumed.
	5	3	30	5·20– 5·30	13·3±0·4	
			Mean value of Φ . .		13·1±0·3	
H	1	6	35	28·3 –35·7	14·5±0·5	2·98
	2	6	45	24·9 –33·6	14·2±0·3	2·95
			Mean value of Φ . .		14·3±0·3	
K	3	5	. .	37·1	14·3	2·98
	3	4	. .	29·4	13·6	2·94
	3	6	. .	30·6	14·7	2·92
	4	6	. .	30·6	11·7	2·95
	5	6	. .	29·7	14·1	2·97
			Mean value of Φ . .		13·7±0·3	

on the positions of outer orders. Accordingly this order was ignored in calculating Φ. The effect is shown in fig. 2. The lower curve represents $P_r \sin^2 \theta'$ plotted against n^2 for the cleavage face and the upper for an etched surface. Galena etched in concentrated hydrochloric acid gave diffraction patterns corresponding to a cross-grating, and showed no refractive index effect. Pl. XIII. figs. 4, 5, & 6, taken along the (001), (011), and (013) azimuths respectively, show this kind of pattern. P-lines were always present, however, in the etched surface pictures.

The mean value of Φ obtained from the H and K methods 14·1 volts, agrees fairly well with that found by Dixit ‡, 12·5 volts.

* *Phys. Zeit.* xxxiii. p. 510 (1932).
† Proc. Phys. Math. Soc. Jap. xvi. p. 95 (1934).
‡ Phil. Mag. cviii. p. 980 (1933) and cxiii. p. 732 (1934).

Pyrites.—The surfaces of this substance were less perfect than those of substances already mentioned. The H and K methods of determining Φ gave results in good agreement with one another. The P-lines due to

Fig. 2

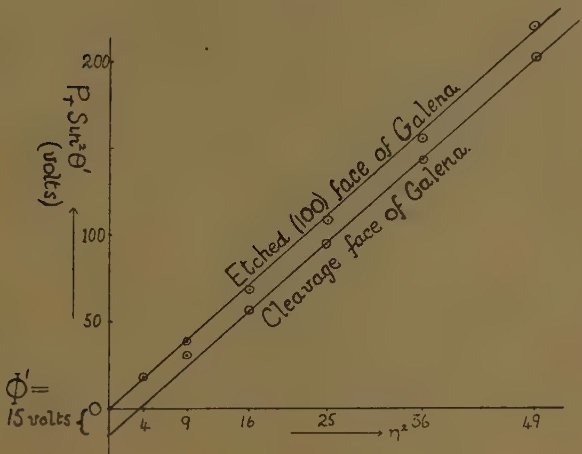


TABLE VII.

Method.	Specimen number.	No. of orders.	No. of readings.	Voltage range kv.	Φ (volts).	d (200) in Å. (calc.).
L	1	1	15	5·05– 5·90	6·2±0·4	2·70 (assumed).
H	1	5	16	30·9 –31·7	3·9±0·4	2·76
	2	4	14	30·6 –32·2	3·9±0·5	2·76
			Mean value of Φ..		3·9±0·3	
K	1	5	..	29·4	3·3	2·76
	2	4	..	32·0	3·9	2·72
			Mean value of Φ..		3·6±0·5	

those planes parallel to the surface were rather broad. Table VII. shows the results obtained for Φ.

It will be seen that in the low voltage range only one order was obtained, the 4th. The value here of Φ is therefore of little importance, beyond showing fair

agreement. The mean value of Φ obtained for P$\sim$30 kv. was 3·8 volts. Dixit has obtained a value of 5·1 volts.

Pyrites etched in concentrated nitric acid showed the cross-grating effect in a manner similar to that of etched galena.

Stibnite.—Only two good specimens were obtained. The effects recently investigated by Emslie * were observed and many of his measurements on the lattice constants were repeated, good agreement with his mean values being obtained. Further work is proceeding on the one dimensional effect found by him.

Yamaguti † found difficulty in assigning order numbers to the diffracted spots using Gottfried's value of $2d$, 11·48 Å., but no difficulty with Ooe's value of 11·26 Å. This same difficulty has been experienced here. The spacing has

TABLE VIII.

Method.	Specimen number.	No. of orders.	No. of readings.	Voltage range kv.	Φ (volts.)	$2d$ in Å (calc.).
L	1	2	14	47·0–5·70	12·7±0·6 or 14·3±0·5	11·48 (assumed). 11·26 is assumed
H	1	5	28	27·4–32·9	14·7±0·2	11·2
K	2	3	..	34·1	15·0	11·2

been calculated, Table VIII., and will be seen to agree much more closely with Ooe's than Gottfried's. Whereas Yamaguti obtained only the 5th, 7th, and 9th orders, in the present work the 6th and 8th have also been observed. Yamaguti's value of Φ, 11·9 volts, is much lower than that calculated here, and no explanation can be offered to account for the difference.

Unfortunately all attempts to obtain a cross-grating pattern from etched stibnite have failed, so that the spacing could not be checked in that manner. No reason can be given for this apparent discrepancy in the spacing.

Fluorspar.—This specimen, which cleaved along the (111) face, was investigated by the three methods, and the results are given in Table IX. The K method could be made to give six orders, but the intensities of the 5th and

* Phys. Rev. xlv. p. 43 (1933).

† Proc. Phys. Math. Soc. Jap. xiv. p. 1 (1932).

6th were always small. This is approximately what would be expected from considerations of the relative positions of the calcium and fluorine atoms in the crystal lattice. Because of this lack of intensity in two of the orders most suitable for measuring Φ by the H method, this method was found rather unsatisfactory. The value of d (111) calculated from X-ray work is 3·14 Å.

Raether *, in a recent paper, has quoted a value of Φ for fluorspar, 15 volts. This appears to be the only previous determination.

Calcite.—Cleavage faces of this substance were used and Φ measured. The results are given in Table X. Yamaguti's difficulty of assigning correct order numbers

TABLE IX.

Method.	Specimen number.	No. of orders.	No. of readings.	Voltage range kv.	Φ (volts).	d (111) in Å. (calc.).
L	3	3	16	49·0 –5·40	13·7±0·3	3·13
	4	2	27	4·62–5·80	13·0±0·3	3·14
			Mean value of Φ..		13·3±0·2	
H	4	3	12	29·0 –31·2	11·6±0·8	3·18
K	1	4	..	30·6	12·1	3·13
	2	4	..	33·2	11·6	3·11
	3	5	..	28·0	11·6	3·15
	5	4	..	30·6	13·2	3·10
			Mean value of Φ..		12·1±0·3	

to the diffracted spots with P∼30 kv. was not experienced here, except in so far as the 3rd order usually gave a value of Φ slightly higher than that obtained from the 4th, 5th, 6th and 7th orders, which were also observed. For this reason the 3rd order value was ignored in calculating Φ. Never more than three orders appeared simultaneously strong enough to measure in the K method, but here no anomaly was observed with the 3rd order. The L method showed two features, absent for the other specimens. The 3rd order gave an exceptionally high value of Φ, assuming d=3·03 Å., the 4th and 5th agreeing quite well. Moreover, the value of Φ appeared to increase slightly with time, though it is doubtful whether much importance should be attached to this point, as the results are hardly

* *Zeit. f. Phys.* lxxxvi. p. 82 (1933).

accurate enough to justify it. The large electronic current and the good insulating properties of calcite probably account for the large value of Φ obtained from the 3rd order, which has been ingored in calculating Φ for the L method as well as for the H method.

The mean value of Φ for P$\sim$30 kv., 12·9 volts, agrees well with those of Yamaguti * and Shinohara †, 12·4 and 13·8 volts respectively.

Gypsum.—A perfectly flat cleavage face of this specimen could not be obtained, although the surface was always

TABLE X.

Method.	Specimen number.	No. of orders.	No. of readings.	Voltage range k.v.	Φ (volts).	d in Å. (calc.).	Remarks.
L	1	2	30	3·73–4·29	12·7±0·4	3·03 (assumed)	—
	3	2	21	4·45–4·70	12·5±0·4	..	New.
	3	2	22	4·88–5·10	15·2±0·5	..	One day *in vacuo.*
	4	2	15	5·25–5·55	12·3±0·4	..	New.
	4	2	14	3·53–5·10	13·5±0·5	..	One day *in vacuo.*
	4	2	12	3·92–5·35	13·4±0·7	..	Two days *in vacuo.*
	Mean value for new specimen			.	12·5±0·2		
H	2	4	30	30·0–32·9	13·0±0·4	3·02	
K	3	3	..	34·7	12·2	3·04	

smooth. The K method had therefore to be abandoned. The H method gave satisfactory results. The L method gave two orders only, but these were in agreement with the H measurements. Actually, the low voltage values of $P_r \sin^2 \theta'$ were 0·5 volt less than the high voltage values for orders 9 and 10. This corresponds to a value of Φ half a volt higher for the L method than the H. The H values are given in Table XI. X-ray work has shown d=7·58 Å.

The mean value of Φ agrees fairly well with that of Yamaguti, 9·1 volts.

* Proc. Phys. Math. Soc. Jap. xiv. p. 1 (1932).
† Sc. Pap. I. P. C. R. xviii. p. 315 (1932).

TABLE XI.

Specimen number	Order nos. obtained.	Voltage kv.	Φ (volts).	d in Å. (calc.).
1	9, 10, 11, 12, and 13	32·9	8·0	7·64
2	6, 7, 8, and 9	25·9	7·7	7·68
		Mean value of Φ..	7·9±0·3	

Discussion.

The values of Φ obtained for the different substances for the two ranges of P are given in Table XII. The mean values given for the H range include both the H and K measurements. The zincblende values combine the results for the S. and P. specimens.

TABLE XII.

Substance.	Value of Φ (volts) for L range.	Value of Φ (volts) for H range.	d in Å. (calc.) H method.	d in Å. from X-ray data.
Rock Salt	6·0±0·2	7·2±0·2	2·80	$2{\cdot}814_4$
Zincblende........	12·1±0·5	12·6±0·15	1·89	$1{\cdot}91_5$
Galena	13·1±0·3	14·1±0·3	2·96	2·98
Pyrites	(6·2±0·4)	3·8±0·3	2·75	2·70
Stibnite	12·7±0·6 14·3±0·5	14·7±0·2	11·2	11·48 or 11·26
Fluorspar	13·3±0·2	11·9±0·3	3·15	3·14
Calcite	12·5±0·2	12·9±0·4	3·02	3·03
Gypsum	8·5	7·9±0·3	7·66	7·58

These results have already been discussed separately as far as agreement with other investigators is concerned. No startling differences have been observed, although the agreement has varied from substance to substance. However, I think little importance can be attached to these differences. Whether or not the different degrees of purity of the specimens used by different workers affect to any appreciable extent the value of Φ is a doubtful point. The present results have shown for zincblende, a substance which cannot be obtained in the pure state, that this effect does not cause variations of Φ of more than a volt.

A little might be said here with regard to the H and K methods. As far as accuracy is concerned, the K method is much the better, providing the specimen is capable of giving at least three orders and the P-lines are sharp: accurate knowledge of the voltage is not required if the spacing is known, which it almost invariably is. The only serious inaccuracy in the K method lies in the determination of the position of the undeviated beam. The spot made by it on the photographic plate was rather large, being about 1·2 mm. in diameter, and not all of it was ever diffracted at once. Consequently it became necessary to know the position of the centre of that portion used, and this could not be done with more than 0·1–0·2 mm. accuracy. This factor undoubtedly contributes to the small discrepancies obtained between different values of Φ for the same specimen. More consistent values of Φ would be obtained were an anode hole of diameter 0·15 mm. used. The advantages of the H method lie in the larger number of orders obtainable and in the fact that the method is applicable to specimens not capable of giving sharp P-lines. The disadvantages include the effect found by Yamaguti for low order numbers.

The more important feature of Table XII. concerns the agreement between the H and L values of Φ. For rock salt and galena the L values appear to be definitely lower than the H: for zincblende, calcite, and gypsum there is little difference, while fluorspar gives an L value rather higher than the H. If, in fact, the same quantity has been measured, these small differences must be accounted for either by casual or systematic errors or by the fact that the conditions have not been quite the same for the two sets of measurements. Considerations of the probable error rule out any possibility of assigning the small differences to casual errors for rock salt, galena, and fluorspar. However, there may have been systematic errors introduced by the observer, tending always to increase or decrease Φ in any special range of P. The determination of the position of the undeviated beam may have caused an error of this type, hardly large enough, however, to account for some of the observed differences of Φ. The second possibility concerns those conditions over which the experimenter has little control.

Any gas layers which might be adsorbed on the surface

of the specimens would be treated differently by the two sets of electrons. Much more of the gas would be removed by bombardment with the faster electrons than with the slower, but in either case a surface layer would most probably remain. It is not known in what manner adsorbed gas layers influence the value of Φ, but any change in this value would vary with P.

Inelastic collisions resulting from the electron bombardment will cause small local heating on the surface of the specimen, but a rough calculation shows that the rise in temperature will never be large enough to cause any detectable change in Φ. Heating effects, therefore, are not responsible for the slightly differing values of Φ.

One effect which must be considered in the case of the very good insulators is that of small positive charging up. It has been shown * that for fairly small angles of incidence specimens attain a positive charge, the slower electrons being capable of this effect at larger angles than the faster. A simple calculation shows, however, that for the specimen surface to be at a potential of 1 volt above that of its immediate surroundings, the undeviated beam would be deflected through an appreciable angle were the crystal face brought almost into it. The least detectable charging up would consequently be much smaller than this, probably less than 0·1 volt.

But charging up, even on a small scale, may help the formation of a double layer on the cleavage face, slowly moving negatively charged ions being attracted by the surface. Such a double layer will cause the Laue spots to be drawn inwards in a manner similar to that of a true refractive index, and the measured value of Φ will no longer represent the true mean inner potential. Since the charging up will vary with P, it is reasonable to suppose that the effect of a double layer on the measurements will also vary with the primary voltage. The difference observed in the values of Φ for galena, which is as large as any, cannot be explained in this way, however, as this substance is quite a good conductor of electricity.

The different penetrating powers of the 30 kv. and 4 kv. electrons will influence to a small extent the measured value of Φ, and in view of Yamaguti's † work,

* An account of this work will be found in 'Conduction of Electricity through Gases.'—Vol. II. by J. J. and G. P. Thomson, p. 183.

† Proc. Phys. Math. Soc. Jap. xvi. p. 95 (1934).

the less the penetration, the smaller the value of Φ observed. It should be borne in mind, however, that although the slower electrons penetrate less for any fixed angle, the angles actually used with the lower values of P have been throughout, larger.

Taking these effects into consideration, it should be inferred from the present results that while a mean inner potential does exist, its measured value is susceptible to small changes, resulting from causes already mentioned. It is worth while here to discuss the true significance of the term "mean inner potential." The presence of the primary electron in or near the surface of the crystal will disturb the charge distribution of the atoms comprising the lattice, and since Φ is essentially a function of this distribution its value will be changed by any disturbance of the charges. Thus it is impossible to measure the "true mean inner potential" inside an undisturbed crystal by the present methods, though it is doubtful whether the difference between the "true" value and that measured here is appreciable. In the same way it is possible to see that the measured value should vary with the azimuth, since the disturbance to the charge distribution is by no means identical for two different directions of the primary beam. However, no effect of this kind has been obtained in the present work, which suggests that any variations of Φ with azimuth are less than a volt.

Bethe's equation $\mu=\sqrt{1+\Phi/\mathrm{P}}$ has been found to hold except for very small angles of incidence, and Φ has been found from this relation to have almost the same value, whether P be 30 kv. or 3 kv. The small differences observed in the cases of rock salt, zincblende, and galena (those substances most carefully investigated) are probably due to the different penetrating powers of the two sets of electrons.

The etched crystals have given diffraction patterns consisting both of extended arrays of spots and of P-lines. These spots can be explained equally simply in two ways. Firstly, we may assume the crystal to be slightly imperfect, when its diffraction pattern would be the same as that of a perfect crystal rotated during exposure to give a variable angle of incidence *. Secondly, we may say that

* See Germer, Phys. Rev. xliv. p. 1012 (1933).

the crystal is indeed perfect, and that the primary beam of electrons is diffracted while traversing the small projections on the surface, the resolving power in the forward direction of the beam being very poor. This is a case of two dimensional diffraction and corresponds to the cross-grating. The P-lines, on the other hand, can be explained only by assuming a perfect crystal and three dimensional diffraction. The first explanation of the spots would require the specimen surface to be imperfect to 1° or more. Such a degree of imperfection of the crystal would forbid any P-lines appearing, so that for a deeply etched specimen it becomes difficult to reconcile this explanation with the actual appearance of P-lines. However, no such difficulty is introduced if we assume the spots to be due to two dimensional diffraction, those satisfying the 3rd Laue condition naturally being the most intense.

Summary.

The mean inner potentials, Φ, of a number of single crystals have been measured, the primary electrons having energies either of about 30 kv. or about 4 kv. The measurements have shown that the low and high ranges of voltage yield very nearly the same values of Φ. Possible reasons for small differences have been discussed. Ethed cleavage faces have also been examined.

Finally, I should like to take this opportunity of thanking Prof. G. P. Thomson, who suggested the use of the slower electrons, for much helpful advice and discussion.

LXVI. *Report on Measurement of Ionospheric Heights at Calcutta during the Polar Year* 1932–33. *By*

H. Rakshit, *D.Sc., Khaira Research Student, Calcutta University.**

[Plate XIV.]

Introduction.

THE work reported in this paper was undertaken at the request of the Secretary of the Polar Year Subcommission on Radio Work of the U.R.S.I. The

* Communicated by Prof. S. K. Mitra, D.Sc.

measurements called for were: (*a*) equivalent heights of the layers of the ionosphere at fixed hours for one or two frequencies, and (*b*) its ionization contents. The arrangement for determining the ionization content being incomplete at the beginning of the polar year, the only measurements undertaken were on the equivalent heights of the layers by the group retardation method [1]. The wave-length used was 75 metres, and the observations were made on the international days of the first and of the second order. On the days of the first order, *i. e.*, on one day (Wednesday) per month, measurements were made at local midday and midnight. On the days of the second order, *i. e.*, on one day per month displaced by fourteen days from the day of the first order, measurements were taken every hour, local mean time, from Wednesday noon to Thursday noon.

The observations were made at Calcutta (long. 88° 22′ E.; lat. 22° 33′ N.; elevation above sea-level 6 metres); the distance between the transmitter and the receiver being only 7 km., the incidence of the exploring waves on the ionosphere was substantially vertical.

Experimental Arrangements.

Some preliminary investigations on measurements of equivalent layer heights by the group-retardation method were made during the latter part of 1931 and the beginning of 1932, a report on which has already appeared in this Journal [2]. Before beginning the polar year observations certain improvements were effected in the transmitting and the receiving systems, which are briefly described here.

The transmitting aerial was of the half-wave horizontal dipole type, and lay approximately in the N.–S. direction. The diode, formerly used as variable grid-leak (*vide* Phil. Mag. ser. 7, xv. p. 22, fig. 1) at the transmitter, was replaced by a wire-wound fixed resistance connected in series with a variable one. This effected a great improvement on the constancy of the pulse frequency at the transmitter.

On the receiving side the oscillograph of Standard Telephones and Cables, Ltd., was replaced by a Cossor C-type cathode ray oscillograph, working at a gun voltage of 2500 for photographic recording and 1000 for visual

observations. The reduction in the sensitivity of the oscillograph, due to high gun voltage, necessitated the addition of a further stage of low-frequency amplification for the receiver and an amplification of the output stage from the 200 volt neon tube oscillator. These modifications greatly improved the linearity of the time base, which was, however, calibrated as before for accuracy of measurement.

The observations were all made on a frame aerial permanently located at a distance of 7 km. to the south of the transmitter. Measurement of the time interval between the direct signal and the echoes was made almost exclusively by visual observations.

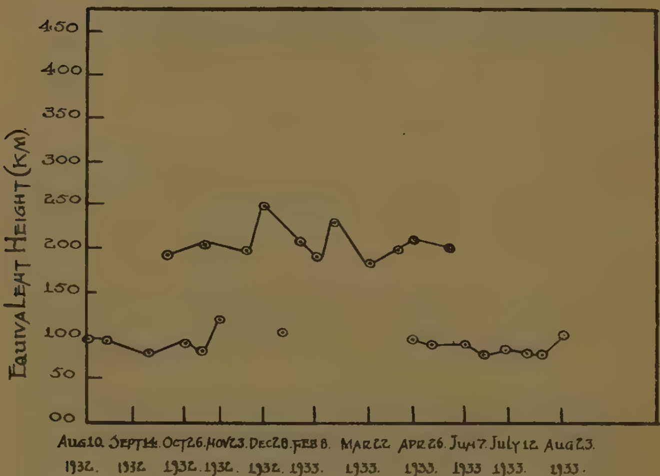
Photographic records of echo patterns were very frequently taken; some typical ones amongst these are reproduced in Pl. XIV. figs. (*a*), (*b*), (*c*), (*d*).

Midday and Midnight Values of the Equivalent Heights of the Ionospheric Layers during the Polar Year (fortnightly records).

The two pairs of curves, figs. 1 (*a*) and 1 (*b*), give the midday and midnight values of the equivalent heights of the conducting layers throughout the whole year. Fig. 1 (*a*) shows that at midday the E-layer is more often penetrated by 75 m. waves in winter than in summer, indicating that the noontime ionization of the E-layer is greater in summer than in winter, as is naturally to be expected, since the solar radiation (ultra-violet or corpuscular) is the chief factor causing ionization of the upper atmosphere. The figure indicates that on certain occasions, *e. g.*, on Nov. 9 and 23, Jan. 25, Feb. 22, and Apr. 26, simultaneous reflexions were obtained from both the E- and the F-layers. It can therefore be concluded [3] that the average noontime ionization of E-layer on each of these dates was equal to that corresponding to a critical penetration frequency of 4 mc./sec. [It should, however, be mentioned here that, although the figure apparently indicates simultaneous reflexion from both the layers on each of these dates, actually this was not the case; for example, on Nov. 23 only E-reflexion was obtained, and on Nov. 24 only F-reflexion; on Jan. 25 only F, and on Jan. 26 only E; on Feb. 22 only F, and on Feb. 23 only E. On Nov. 9 and

Apr. 26, however, E and F were simultaneously obtained. Anyhow, if at noon on any one day E alone is obtained and on the next day only F, it can be concluded that the average penetration wave-length nearabout this date is approximately 75 metres.] Fig. 1 (*a*) also clearly depicts the seasonal variation of noontime ionization of the E-layer. During August and September only E-reflexion was obtained; with the progress of the year the E-ionization diminished, and in October and

Fig. 1 (*a*).



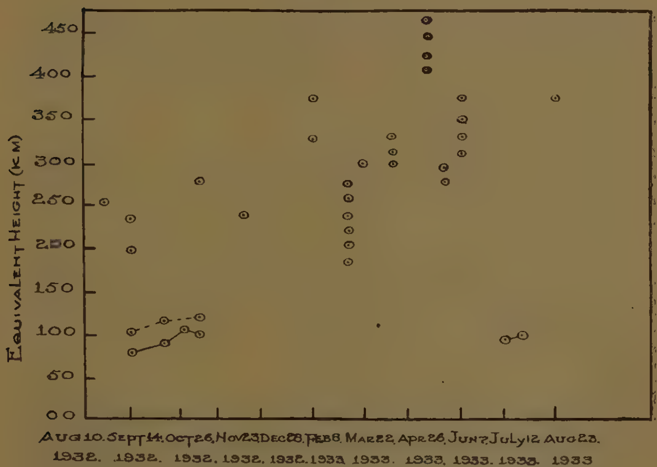
Midday values of the equivalent heights of the E- and F-layers throughout the polar year 1932–33.

November simultaneous reflexion from both the E- and the F-layers was obtained. With further progress, during the cold days E-ionization still diminished, and only F was obtained. From April the ionization gradually increased, and during April and May simultaneous reflexion from both the layers was obtained. The increase of ionization continued until in summer it was so great that the 75-metre waves were unable to penetrate the E-layer, and only E-reflexion was obtained.

Fig. 1 (*b*) depicts the result of the measurement of the equivalent height of the reflecting layer at midnight.

It is seen that during the major portion of the year the E-layer was penetrated at midnight by the 75-metre waves. Sometimes the F-layer was also penetrated. On certain occasions an echo was split up into some four, five, or more components. Each of these components naturally gave a corresponding value of the "equivalent height," as has been shown in the figure, corresponding to 8.3.33, 12.4.33, and so on. With the present state of our knowledge about the ionosphere no definite

Fig. 1 (*b*).



Midnight values of the equivalent heights of the E- and F-layers throughout the polar year 1932–33.

meaning can be attached to such multiple values of "equivalent height." The complex types of splitting of the 75-metre waves under consideration were observed only during the night. It is found that about an hour after the appearance of such split echoes the signals fail to return to the earth, indicating that the ionosphere has been penetrated. These facts indicate that such complexities are due to the ionization gradient in the ionosphere being very small. Under such circumstances the two components of magnetically split up [4] waves might again be split up, and so on, producing

the large number of components of one single echo. Complex echoes are also observed during the pre-dawn period—these might also be due to the irregularity and ill-definition of the conducting layer. In such a case the receiver might catch reflexions from various points of the layer, and the equivalent paths of these echoes might be widely different, thus apparently indicating as if one single echo had been split up into so many components.

Diurnal Variation of the Equivalent Heights of the Ionospheric Layers during the Polar Year (*monthly records*).

Curves (a) to (m), fig. 2, depict the results of twenty-four hour measurements of height, once every month, throughout the whole polar year. A general inspection of all the curves shows :—

(i.) The height of either layer is in general a minimum during the midday hours and a maximum just before ground sunrise.

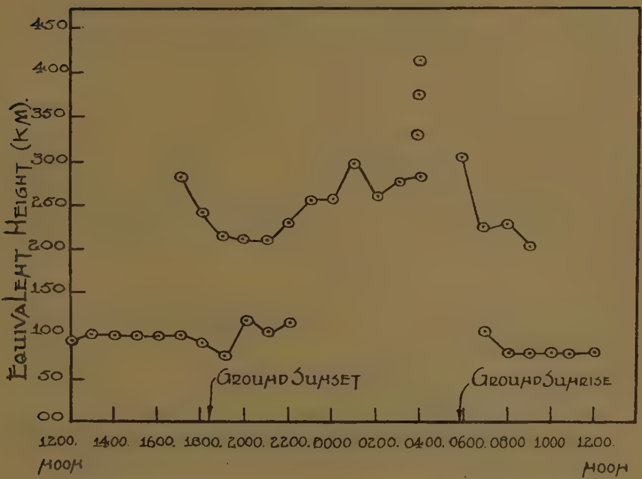
(ii.) The percentage variation of the height of the E-layer is much smaller than that of the F-layer, indicating that the E-layer is comparatively less extended vertically than the F-layer.

(iii.) There is a sudden drop in the recorded height sometime between 1700 Hm. and 2100 Hm. The drop is sudden, but the hour at which the drop occurs is not the same for all the days. The echoes also generally increase in strength at this time. This is most probably due to the so-called "evening concentration" of ionization, which is supposed to occur daily a few hours after sunset [5].

(iv.) During the twenty-four hour period of observations with 75-metre waves the E-layer is penetrated for a greater number of hours in winter than in summer.

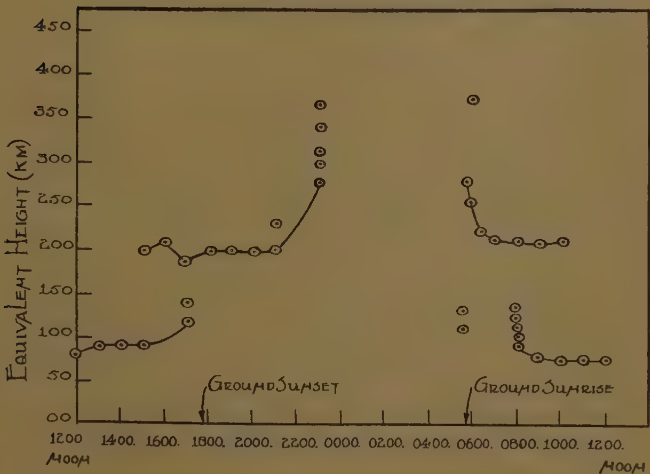
(v.) The splitting of an echo by the influence of the earth's magnetic field is more frequent, and the interval between the components of a doublet much longer in the case of the F-layer echoes than in the case of the E-layer ones. This indicates that the gradient of ionization is much greater in the E-layer than in the F-layer.

Fig. 2 (*a*).



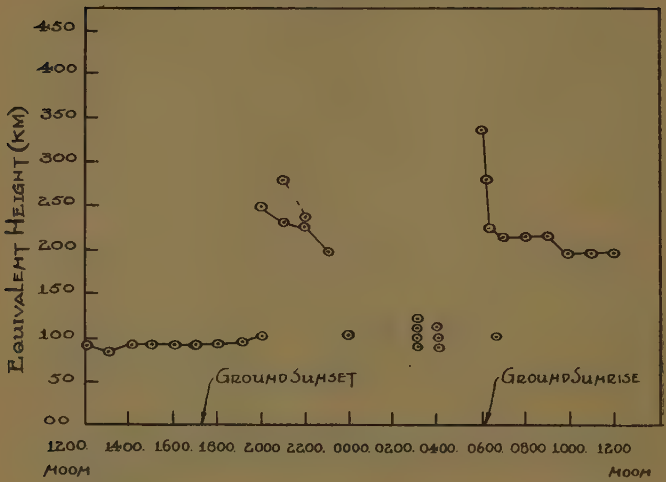
Diurnal variation of the equivalent heights of the ionospheric layers on August 24–25, 1932.

Fig. 2 (*b*).



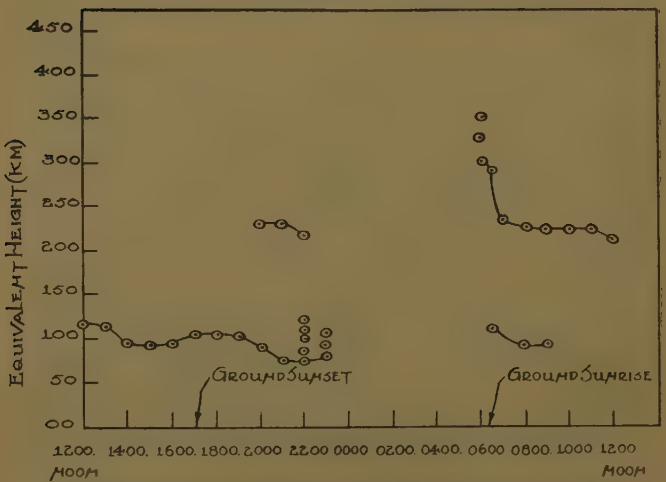
Diurnal variation of the equivalent heights of the ionospheric layers on September 28–29, 1932.

Fig. 2 (*c*).



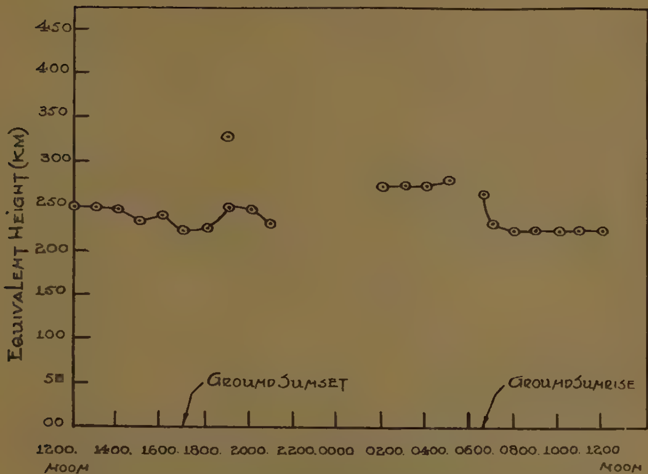
Diurnal variation of the equivalent heights of the ionospheric layers on October 26–27, 1932.

Fig 2 (*d*).



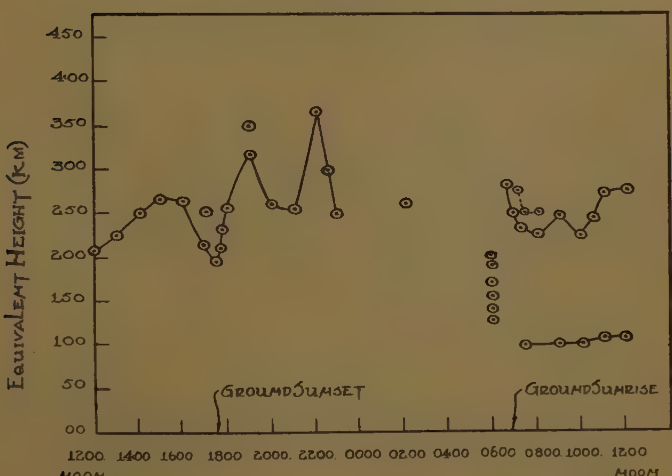
Diurnal variation of the equivalent heights of the ionospheric layers on November 23–24, 1932.

Fig. 2 (*e*).



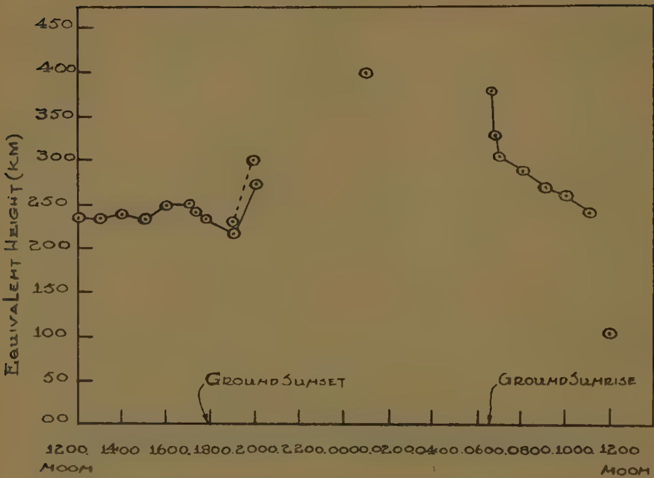
Diurnal variation of the equivalent heights of the ionospheric layers on December 28–29, 1932.

Fig. 2 (*f*).



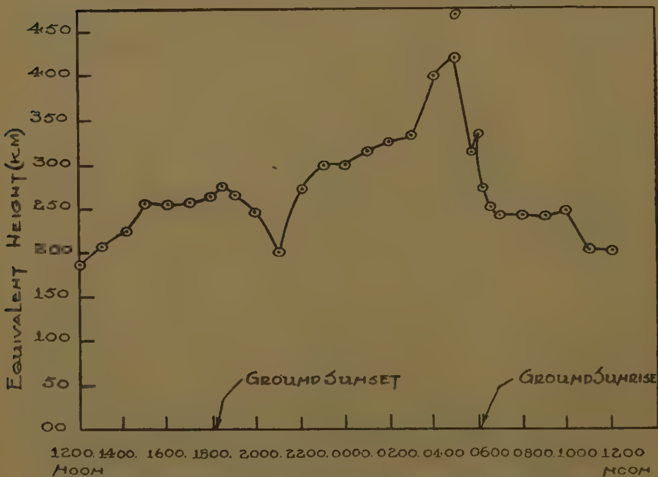
Diurnal variation of the equivalent heights of the ionospheric layers on January 25–26, 1933.

Fig. 2 (*g*).



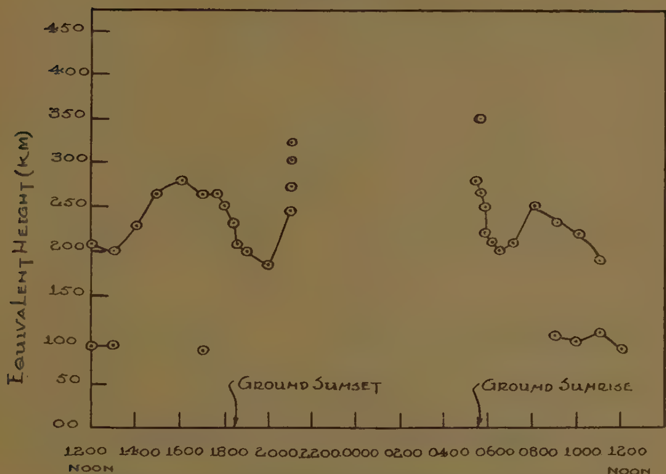
Diurnal variation of the equivalent heights of the ionospheric layers on February 22–23, 1933.

Fig. 2 (*h*).



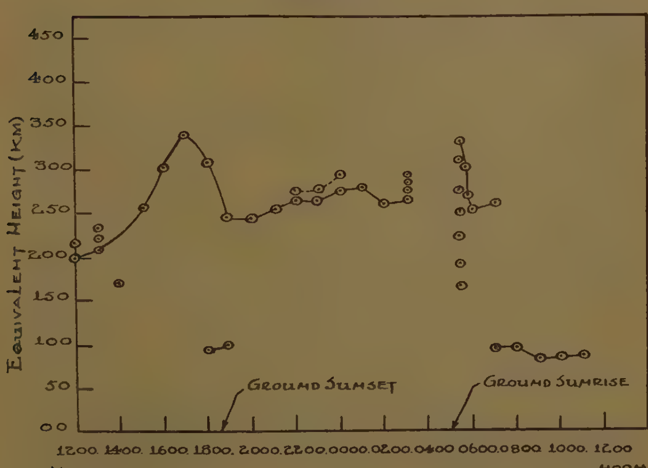
Diurnal variation of the equivalent heights of the ionospheric layers on March 22–23, 1933.

Fig. 2 (*i*).



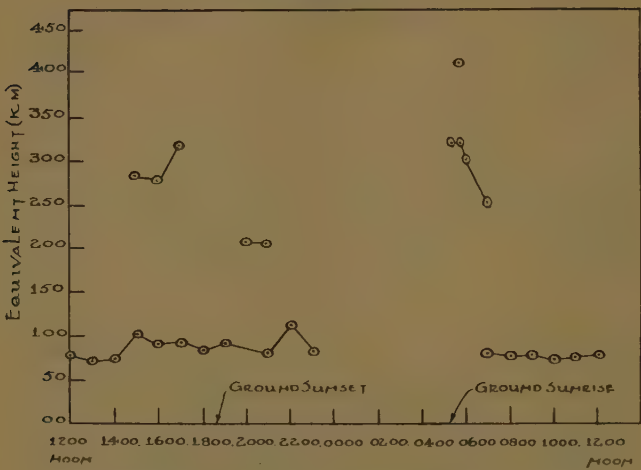
Diurnal variation of the equivalent heights of the ionospheric layers on April 26–27, 1933.

Fig. 2 (*j*).



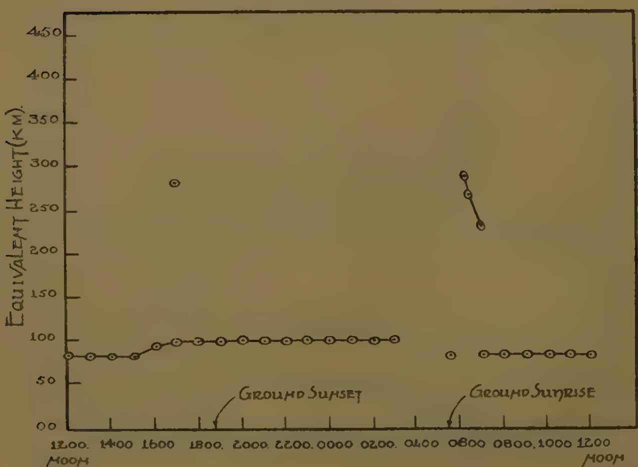
Diurnal variation of the equivalent heights of the ionospheric layers on May 24–25, 1933.

Fig. 2 (*k*).



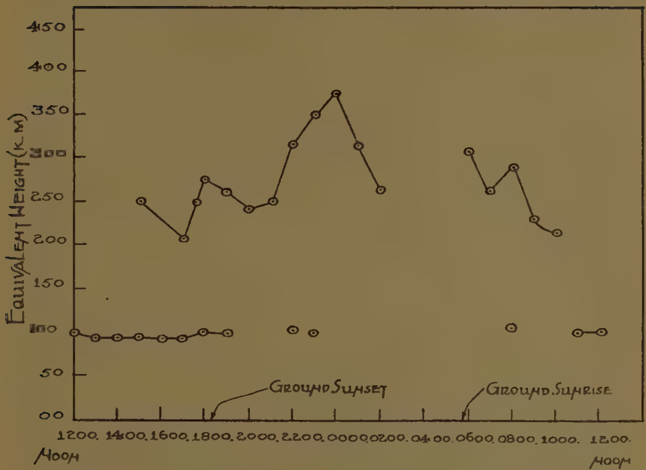
Diurnal variation of the equivalent heights of the ionospheric layers on June 21–22, 1933.

Fig. 2 (*l*).



Diurnal variation of the equivalent heights of the ionospheric layers on July 26–27, 1933.

Fig. 2 (*m*).



Diurnal variation of the equivalent heights of the ionospheric layers on August 23–24, 1933.

(vi.) As the average height of F-layer increases with the progress of the night, the interval between the components of a doublet also increases. This signifies that the gradient of ionization diminishes with increase of height. This also explains the fact that splitting is more frequent during the night than during the day, since the signals are reflected during the daytime from lower levels of the F-layer, where the gradient is rather large.

(vii.) An inspection of curves in figs. 2 (*a*), 2 (*b*), 2 (*d*), etc. shows that the appearance of multiply split echoes is followed up immediately by complete penetration of the F-layer. It seems as if the upwardly directed waves enter up the F-layer step by step until it is completely penetrated.

These are some of the general features which are common to all the curves.

The strength and the number of the echoes were closely observed and recorded during the measurements of the height. The records show that the echoes from either layer considerably gain in number and intensity during the sunrise and the sunset periods. The intensities

also undergo more rapid fluctuations during these periods than at other times.

We now proceed to give a detailed description of each curve separately.

Curve 2 (*a*) (*Aug.* 24–25, 1932).—From 1200 Hm. to 1600 Hm. only E-layer reflexions were obtained. From 1700 Hm. to 2200 Hm. simultaneous reflexions were obtained from both the E- and the F-layers, indicating that the average ionization content of the E-layer was during this period nearly equal to that corresponding to a critical penetration frequency of 4 mc./sec. From 1700 Hm. to 1900 Hm., however, the height of the E-layer (F-layer also) gradually diminished, naturally leading to the conclusion that the ionization also increased, a fact which makes it rather difficult to explain the simultaneous reflexion of waves from both the layers, because, as there was no simultaneous reflexion before 1700 Hm., an increase of E-layer ionization would mean a greater difficulty for the waves to penetrate this layer after 1700 Hm. than before. Again, during this period (1700 Hm. to 1900 Hm.) the intensity and the number of echoes were much greater than before; but, since collisional friction is greater at lower levels than at higher ones, an increase of ionization causing reflexion from lower layers would necessitate a reduction of echo intensity—a fact which is contrary to the observed results. If, however, the real cause of the feebleness of the echoes before 1700 Hm. be assumed to be due chiefly to electron limitation, and not to collisional absorption, then the increase of strength of the echoes may be taken to support the conclusion of an increase of ionization from 1700 Hm., due to the arrival of electrons. Based on this point of view the simultaneous reflexion of signals from both the E- and the F-layers from 1700 Hm. can also be supported if we assume that the absence of F-reflexion before 1700 Hm. was due, not to the difficulty of penetration for the 75-metre waves, but due to poor reflexion on account of electron limitation.

At 0400 Hm. the echo was split up into a large number of components. It is seen that, as mentioned previously, the F-layer was completely penetrated after the appearance of this complex splitting, and at 0500 Hm. there was no echo. After this the echo was first observed at 0600 Hm. The corresponding effective height was quite

large, which then rapidly fell, as is usual at sunrise. The F-reflexions were, as usual, strong from about 0615 Hm. to 0650 Hm., but the E-reflexions were strong between 0800 Hm. and 0900 Hm.

Curve 2 (*b*) (*Sept.* 28–29, 1932).—From 1200 Hm. to 1400 Hm. only E was obtained. At 1500 Hm. E and F were simultaneously received, indicating a diminution of E-layer ionization. This was further supported by the absence of E-reflexion at 1600 Hm. and a large E-layer height at 1700 Hm., after which no E was received during the whole of the night. At 2100 Hm. the F-echo was split up into two, and at 2300 Hm. into a number of components due presumably to a large reduction of ionic concentration in the layer. The ionization further diminished with the advancement of the night, and from midnight to about 0540 Hm. no echo was recorded, the signals penetrating both the E- and the F-layers. The first F-reflexion was then observed at 0550 Hm., and was very weak. At 0600 Hm. the first-order echo F_1 was split up into two, both components being weak; the second-order echo F_2 was, however, not split up, one of the components being most probably absorbed.

After 0600 Hm. the height rapidly diminished, as usual. From 0600 Hm. to 0620 Hm. the intensity and the number of echoes were much greater than before and after. The ionization having increased with the progress of the day, E- and F-reflections were simultaneously obtained from 0800 Hm. to 1000 Hm. After this E alone was present. The E-reflexion first observed at 0800 Hm. consisted of a number of components due presumably to poor ionization.

Curve 2 (*c*) (*Oct.* 26–27, 1932).—From 1200 Hm. to 2000 Hm. only E was received. At 1900 Hm. E was split up, and at 2000 Hm. the ordinary component was probably lost, due to further reduction of ionization. On account of this reduction, E-layer height increased at 2000 Hm. and a simultaneous F-echo was received. From 2100 Hm. only F was recorded; at 2100 Hm. and 2200 Hm. F was split up into two, but the interval between the components was less at 2200 Hm. than at 2100 Hm., due to an increase of ionization, as is evident from a further lowering of F-layer height at 2300 Hm. At midnight a faint E was alone obtained, and there was no echo at 0100 Hm. and 0200 Hm. At 0300 Hm.

and 0400 Hm. a very faint split up E was observed, and again no echo between 0500 Hm. and 0600 Hm. The first echo next received was from the F-layer at 0600 Hm. with a corresponding large effective height, which rapidly diminished, as usual.

Curve 2 (*d*) (*Nov.* 23–24, 1932).—From 1200 Hm. to 1900 Hm. E alone was obtained. At 2000 Hm. and 2100 Hm. F was received simultaneously with E. At 2200 Hm. and 2300 Hm. E was split up, and after this no echo was observed until 0550 Hm., when a split-up F was recorded, giving a large value of the effective height. This height then rapidly diminished, as usual.

Curve 2 (*e*) (*Dec.* 28–29, 1932).—No E was recorded within the twenty-four hours. F was received from 1200 Hm. to 2100 Hm., being split up into two components at 1900 Hm. After 2100 Hm. no echo was obtained until 0200 Hm., when two very weak echoes were received. At 0500 Hm. the number of echoes was five, but all of them were weak. From 0600 Hm. no echo was recorded until 0635 Hm., when echoes were again obtained. The height was then large, but rapidly diminished, as usual.

Curve 2 (*f*) (*Jan.* 25–26, 1933).—From 1200 Hm. to 2300 Hm. only F was obtained, with three exceptions at 1400 Hm., 1500 Hm., and 1600 Hm., when one weak E was simultaneously recorded. The variation of the F-layer height was peculiarly irregular, a number of ups and downs being recorded. The minimum height was obtained just before ground sunset. After 2300 Hm. echoes were absent during the night, with an exception at 0200 Hm., when F-echoes were received, the first order one (F_1) being moderately strong. This might be due to ionic clouds or such other local causes. The first sharp echo was next recorded at 0640 Hm., giving a large effective height. At 0700 Hm. the height diminished, but the echo was split up. At 0725 Hm. a simultaneous E was received, and from 0900 Hm. to 1200 Hm. E and F were always simultaneously present, both becoming very weak with the progress of the day.

Curve 2 (*g*) (*Feb.* 22–23, 1933).—No E was recorded within the twenty-four hours except at 1200 Hm. on February 23. At 1900 Hm. the F-echo was split up into two components, the interval between which increased

at 2000 Hm. After this echoes were absent during the whole of the night, with an exception at 0100 Hm., when a very weak F was recorded with a correspondingly very great height. The first echo was next observed at 0635 Hm. from the F-layer, the height being very large. This then diminished, at first very rapidly and then gradually. At midday F was absent and a moderately strong E was recorded.

Curve 2 (*h*) (*Mar.* 22–23, 1933).—No E was recorded during the twenty-four hours. From 1200 Hm. the height at first increased gradually up to 1824 Hm.; after this it fell rapidly to a minimum at 2100 Hm., and then went on increasing again, reaching a maximum at 0500 Hm., when the echo was split up into two. After this the height very rapidly dropped, as usual.

Curve 2 (*i*) (*Apr.* 26–27, 1933).—At 1200 Hm. and 1300 Hm. E and F were simultaneously obtained, and from 1400 Hm. to 2100 Hm. only F, with an exception at 1700 Hm., when a moderately strong split-up E was recorded. At 2100 Hm. the echo was split up into a large number of components, and after this no reflexion was received during the night. The first echo next received was at 0524 Hm.; at 0532 Hm. F_1 and F_2 were quite strong and both split up into two; F_3 and F_4 were weak and not split up. The height then rapidly fell to a minimum at 0620 Hm., after which it again increased to a maximum at 0800 Hm. The ionization having increased with the progress of the day, E and F were simultaneously obtained from 0900 Hm. to 1100 Hm., and with further increase of ionization E was alone received at 1200 Hm.

Curve 2 (*j*) (*May* 24–25, 1933).—From 1200 Hm. to 1700 Hm. F was alone received, and during this period the F-layer height continually increased. The height then rapidly diminished to a minimum at 1845 Hm. Betweem 1800 Hm. and 1900 Hm. strong reflexions from both E- and F-layers were simultaneously received. After 1900 Hm., however, E was again absent throughout the night. At 0300 Hm. F was split up into a number of components, and was entirely absent after this. The first reflexion was next observed at 0520 Hm., and consisted of a large number of weak components. At 0528 Hm. the echo (F) was well defined and quite strong; the height was, however, great, and rapidly fell, as usual.

At 0700 Hm. E and F were simultaneously obtained, and after this with increase of ionization E alone was received.

Curve 2 (*k*) (*June* 21–22, 1933).—Regular reflexions were obtained from the E-layer only, the F-echoes being stray ones except during the sunrise period. At 1500 Hm., 1600 Hm., and 1700 Hm. one single F-echo was obtained simultaneously with E. At 1500 Hm. and 1700 Hm. the F-echo was weak, but at 1600 Hm. it was very strong. The interval between the ground-signal and the F-echo at 1600 Hm. corresponded to a path-difference of 2×472 km. This was presumably a so-called "M-reflexion," *i. e.*, the ray had travelled from the transmitter to the F-layer, then back to the E-layer, then again to the F-layer, and finally back to the earth. The height of E-layer at 1600 Hm. was 92 km., and hence that of the F-layer 282 km., as plotted on the curve. At 2000 Hm. E-reflexion was again entirely absent, and as many as eight echoes were received from the F-layer, the first three being very strong. At 1900 Hm. only E's were received, the first-order echo E_1 being very strong. The absence of E at 2000 Hm. and simultaneous E and F (very weak) at 2100 Hm. indicates a sudden reduction of E-layer ionization nearabout 2000 Hm., which was soon replenished. After 2300 Hm. neither E nor F was received throughout the night. Several F's were obtained just after ground sunrise. At 0700 Hm. E and F were simultaneously received, indicating a gradual increase of E-layer ionization. From 0800 Hm. onwards only the E's were present.

Curve 2 (*l*) (*July* 26–27, 1933).—F-reflexions were only present during the sunrise period; at 1700 Hm., however, two weak F's were received along with E. After 0300 Hm. neither E nor F was obtained. From 0520 Hm. to 0530 Hm. E's were received which were not very weak—this was most probably due to some local effect which caused an increase of E-layer ionization. From 0615 Hm. to 0700 Hm. reflexions were obtained in the usual course from the F-layer, simultaneous reflexions from both layers being obtained at 0700 Hm. After this the E-layer ionization increased with the progress of the day, and from 0800 Hm. onwards only E's were obtained.

Curve 2 (*m*) (*Aug.* 23–24, 1933).—At 1500 Hm. E was

absent and only F present. From the record of twenty-four-hour ionization measurement on this day E-ionization was found to have diminished considerably at this time. After this the ionization was replenished, but was again low at 2000 Hm. and 2100 Hm., with the result that E's were absent during this period. After the penetration of both the layers during the early hours of the morning the first trace of echo was obtained at 0600 Hm. from the F-layer. At 0700 Hm. the ionization and height of the F-layer was less than at 0600 Hm. At 0800 Hm., however, E-ionization abruptly increased and that of F diminished, resulting in a simultaneous reflexion from both layers. After 0800 Hm. E-ionization again diminished, as evident from the absence of E-echo. From 1100 Hm. the E-ionization was approximately normal.

Periods of Appearance and Disappearance of Echoes at unexpected Hours. (See figs. 2(*a*) to 2(*m*), pp. 681–687.)

Curve 2 (*a*) (Aug. 24–25, 1932).—During the twenty-four hours there were no unexpected periods of appearance or disappearance of echoes.

Curve 2 (*b*) (Sept. 28–29, 1932).—The split E's from 112 and 130 km. at 0540 were unexpected. Under normal conditions the echoes from F region ought to have persisted longer than actually observed.

Curve 2 (*c*) (Oct. 26–27, 1932).—The echoes from E at 0000, 0300, and 0400 were unexpected.

Curve 2 (*d*) (Nov. 23–24, 1932).—From 2000 to 2300 the E's were very strong and underwent rapid fluctuations. the E's from 2100 to 2300 were rather unexpected under normal conditions, but these can be accounted for on the basis of "evening concentration." The absence of E's from 1000 to 1200 was unexpected.

Curve 2 (*e*) (Dec. 28–29, 1932).—The absence of F's from 2200 to 0100 was unexpected.

Curve 2 (*f*) (Jan. 25–26, 1933).—Four F's at 0200 and a large number of ill-defined faint ones at 0600 were unexpected.

Curve 2 (*g*) (Feb. 22–23, 1933).—The stray F at 0100 was very weak, and as such it cannot be called unexpected.

Curve 2 (*h*) (Mar. 22–23, 1933).—No unexpected appearance or disappearance of echoes.

Curve 2 (*i*) (Apr. 26–27, 1933).—There is a sudden

appearance of E at 1700, indicating an increase of E-ionization.

Curve 2 (*j*) (May 24–25, 1933).—The same phenomenon between 1800 and 1850.

Curve 2 (*k*) (June 21–22, 1933).—There is a break in the simultaneous reflexion from E and F at 1800 and 1900, indicating an increase in the E-layer ionization. The appearance of E alone at 2200 and 2300 was unexpected.

The increase in the E-ionization between the hours 1700 and 1900, as observed in the last three curves, is probably related to the frequent occurrence of thunderstorms—the so-called Nor'westers—during these hours in this part of India.

Curve 2 (*l*) (July 26–27, 1933).—The stray echoes (F_1, F_2) at 1700 were weak and were rather unexpected. The E's from 0520 to 0530 were also unexpected.

Curve 2 (*m*) (Aug. 23–24, 1933).—The weak E's at 2200 and 2300 were rather unexpected. The absence of E's at 0900 and 1000 was also unexpected.

Reflexions below an Equivalent Height of 100 *km.*

No indication of any layer below the normal E region was obtained. From the curves it will be seen that some reflexions were obtained from an equivalent height of 75 to 80 km. This low height is certainly due to the geographical position of Calcutta and to changes of E-layer ionization, but not to the presence of any layer below the E region.

Summary and Conclusion.

This is a report of the observations on the equivalent heights of the ionospheric layers carried out at Calcutta in accordance with the U.R.S.I. Polar Year, 1932–33, programme. The wave-length used was 75 metres (4 mc./sec.), and measurements—by the group retardation method [1]—were made on international days of the first and the second order. Closer observations were taken very frequently during the sunrise and the sunset periods, as the echoes possess many interesting features at these hours.

It has been found that the height of either layer generally attains a minimum value during the midday hours. The recorded height is a maximum at about twenty to thirty minutes before ground sunrise, and then falls

abruptly to the normal daytime value within about one hour after ground sunrise.

Regarding the seasonal variation of height, it may be said that on an average the height of either layer is greater in winter than in summer. The noontime ionization of either layer is greater in summer than in winter.

The observations indicate that the F-layer is more extended vertically than the E-layer.

There is a sudden drop in the recorded height sometime between 1700 Hm. and 2100 Hm.; the drop is sudden, but the hour at which it occurs is not the same every day. The echoes generally increase in strength at this time, due, most probably, to the so-called "evening concentration" [5] of ionization, which is supposed to occur daily a few hours after sunset.

During the night time an echo from the F-layer is frequently found to be split up into a large number of components, due to the influence of the earth's magnetic field [4]. Immediately after the appearance of such splitting the echoes are found to disappear completely. An explanation of this interesting phenomenon is given.

Observations indicate a steeper gradient of ionization density in the E-layer than in the F-layer. It has also been found that the gradient diminishes with increase of height.

In conclusion, it may be said that a single set of observations, as described in this communication, is insufficient for giving any theoretical interpretation of all the observed effects or for drawing definite conclusions regarding the agencies responsible for the ionization of the upper atmosphere. Important conclusions may be expected from a statistical examination of the results of observations made by all the workers co-operating in the polar year programme.

The request for participating in the polar year programme was addressed to Prof. S. K. Mitra by Mr. R. A. Watson Watt, the Secretary. The investigation by the author was undertaken at Prof. Mitra's suggestion, to whom he offers his sincere thanks for constant help and guidance. Thanks are also due to Mr. P. Syam, M.Sc., and Mr. B. N. Ghosh, M.Sc., who helped in the observations by attending to the transmitter whenever required.

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LXVII. *The High-Frequency Glow Discharge.* *By* John Thomson, *M.A., D.Sc., Lecturer in Natural Philosophy at the University of Glasgow* *.

I. *Introduction.*

It is the purpose of this paper to describe and to discuss certain new experiments on the initiation and maintenance of the high-frequency glow discharge with particular reference to a simple electrostatic theory of ionization in a high-frequency alternating electric field [1]. The technique which has been employed is believed by the writer to be the only one which can yield results of theoretical significance, and the reasons for using this technique have therefore been given in some detail. An important result which has been suggested by the experiments is that in the production of the direct-current discharge the ionization of gas molecules by collision with positive ions plays no part. The experiments do not contradict in any way the theory of the discharge mentioned above [1]. Further experiments at higher frequencies are, however, necessary before the value of the hypothesis can be properly estimated.

II. *General Remarks on Previous Investigations.*

According to the means by which they are excited, high-frequency (H.F.) discharges may be placed in one of two categories. The gas at a low pressure may be contained in a tube around which a coil of wire is placed,

* Communicated by the Author.

and the discharge may then be produced by means of the field due to an H.F. alternating current flowing in the coil. Alternatively the coil may be placed inside the tube. This category will be named "Class A Excitation." On the other hand, the discharge may be produced between two electrodes maintained at an H.F. alternating potential difference, and the electrodes may be placed either inside or outside of the tube containing the gas. This category will be named "Class B Excitation."

The experiments of MacKinnon[2], Knipp[3], and others have finally decided the vexed question of the origin of the initial ionization. These investigators have shown that when Class A excitation is employed, the initiation of the glow is due to the electrostatic field of the coil, and that it is only after the discharge has been struck that the effect of the electromagnetic field is felt by the gas ions. They have also demonstrated that the surprisingly bright glow sometimes obtained by method A owes its intensity to electromagnetic action. The electrostatic field alone cannot produce the relatively large ionic currents observed at very low pressures. When Class B excitation is employed, the field between the electrodes is, of course, almost completely electrostatic, so that there is no question in this case of the origin of the ionizing action.

Other investigations of the high-frequency discharge may be roughly divided into two classes—(*a*) those concerned with the initiation and maintenance of the discharge, and (*b*) those concerned with the "plasma electron oscillations"[4, 5] which take place so readily in the highly ionized gas obtained by H.F. excitation. With the latter class this communication is not directly concerned. To class (*a*) belong the investigations of Gutton[6, 7], Kirchner[8, 9], Townsend and his collaborators[10, 11, 12, 13], and Brasefield[14, 15], who have all, using Class B excitation, measured either the striking potential (to be denoted by V_s), the minimum maintenance potential (V_m), or the potential necessary to produce a certain current in the discharge, as a function of the frequency of the H.F. oscillation and of the gas pressure. Smith, Lynch, and Hilberry[16] and Esclangon [17] (among others) have made similar measurements, using Class A excitation.

There are several important matters of experimental technique which must be considered before any opinion may be formed of the theoretical value of these investiga-

tions. Several of the writers mentioned above, notably Gutton and Brasefield, have used external electrodes. Gutton in some of his experiments used plane electrodes fixed to the ends of his discharge-tube ; Brasefield and some of Townsend's collaborators used metallic rings encircling the cylindrical tube and movable along its length. Although such electrodes are useful, and in many cases are more convenient than internal ones, it appears to the present writer that the latter are much to be preferred in a quantitative investigation which aims at a theory of the discharge mechanism. For that reason Kirchner's first experiments probably form a more satisfactory foundation for speculation. The absence of electrodes from the inside of the tube certainly eliminates the surface effects which have proved so troublesome in the case of the D.C. discharge (18). But the glass walls of the tube, having taken the place of the electrodes, store up negative electric charge in a manner determined by the relative mobilities of the positive and negative ions. Moreover, with external electrodes it is impossible to keep the discharge away from the side walls. Hence, in a dry tube containing dry gas, after the first discharge has taken place the charge on the inner surface of the walls is quite unknown ; clearly, it may seriously disturb the electrostatic field produced by the electrodes. The same argument applies to tubes with internal electrodes, if the latter are not well clear of the walls.

A second important experimental condition is the purity of the gas used in the discharge-tube. That the conclusions of several otherwise excellent investigations have been invalidated by impure gas will be shown later from the present writer's experiments. The presence of an impurity not only causes the observations to be useless with reference to the pure gas, but in many cases so alters the results as to render them widely different from those obtained with *any* pure gas and only typical of a certain type of mixture. In this respect it is doubly unfortunate that mercury vapour finds its way so readily into discharge-tubes, for the effect of even a trace of this substance is to change completely the striking and maintenance potentials.

In investigations of D.C. discharges the well-known spurious effects due to the proximity of the walls of the tube to the space between the electrodes have been

eliminated by making the distance between the electrodes relatively small. For example, J. Taylor in some of his most refined work[19] used electrodes 5 cm. in diameter and about 2 cm. apart in a bulb of approximate diameter 10 cm. In the case of H.F. discharges such a procedure is unfortunately almost impossible, for, as will be seen later, unless the frequency of the applied potential difference is of the order 10^8/sec. or greater, the distance between the electrodes must be of the order 10 cm. or greater. Hence the appropriate technique for frequencies of the order 10^6/sec. is not to decrease the gap between the electrodes, but to increase the size of the container, a modification which is not without its difficulties. To exemplify this point mention may be made of the experiments of Gutton, Mitra, and Ylostalo[6] in which the maximum frequency of the H.F. oscillation was $2{\cdot}14 . 10^6$/sec. With frequencies of this order a typical H.F. discharge (which will be defined later) takes place only if the distance between the electrodes is of the order of 50 cm. or more. Needless to say the tube used was smaller.

Summarizing, it may be said that investigations the aim of which is the elucidation of the mechanism of the H.F. discharge are only satisfactory if :—(1) Internal electrodes are used ; (2) the gas is spectroscopically pure, and in particular is free from any trace of mercury vapour ; (3) the distance between the electrodes is greater than $10^6/\nu$ cm., where ν is the frequency of the applied oscillation in cycles per sec. ; (4) the dimensions of the tube are such that " wall effects " are negligible.

III. *Electrostatic Theory of the Initiation of an H.F. Discharge.*

In a previous communication[1] the writer suggested the general conditions which must be satisfied before a gas may be ionized by electronic impact, the electrons being assumed free to move under the action of an alternating field. These conditions were reduced to the inequality

$$E > \frac{Vp}{K} + \frac{2\pi^2\nu^2 K}{pe/m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the meanings of the symbols are as follows :—

E = The maximum or peak value in absolute e.m.u. of the electric field $E \cos 2\pi\nu t$.

ν=The frequency of the alternation in cycles per sec.

V=An individual gas constant in absolute e.m.u. of potential, related to the ionizing potential of the gas.

L=A distance in cm. proportional to the mean free path of an electron in the gas at pressure p, the latter being given in mm. of mercury.

K=The individual gas constant connecting L and p by means of the equation $L=K/p$.

It was assumed in the first instance that the mean free path of the electron is independent of its velocity.

Inequality (1) defines the minimum value of E for which ionization will take place by electronic collision in an infinite volume of gas throughout which the uniform field $E \cos 2\pi\nu t$ is acting. If the frequency of the oscillation is made negligibly small, so that over an appreciable time the field strength remains E, then ionization will occur for $E=V/L$. This simple condition exhibits clearly the nature of the physical assumptions which have been made. The molecules of the gas are supposed to be stationary with respect to the electrons and separated by equal distances, so that an electron travels L cm. between each collision with a molecule. Also, the electrons are assumed to be initially at rest relative to their ionizing velocities.

When a steady potential is applied across a volume of gas, the conditions for ionization by electronic impact are not the conditions for the production of a discharge, since the presence of the electrodes, associated with the unidirectional field, cause the "life" of a gas ion to be exceedingly short. If, however, an H.F. alternating potential is used, and the electrodes are sufficiently far apart, it is possible to adjust both the frequency and the field strength so that many of the ions formed will remain in the gas for a considerable time, oscillating to and fro about an approximately stationary centre. There are still losses due to recombination and diffusion, but the life of an electron may be so prolonged that the conditions for ionization may be the approximate conditions for a discharge. Clearly, if the walls of the tube and the electrodes are at a large distance from the region in the gas under consideration, inequality (1) is then a necessary and sufficient condition for the initiation of a self-sustained glow.

If inequality (1) is satisfied, it is impossible that the amplitude of oscillation of a free electron under the action of the field should be less than L, and therefore it is impossible that ionization should fail to occur in the first half oscillation. This presupposes that the distance between the electrodes in the tube is great enough to allow this oscillation to take place undisturbed apart altogether from the considerations of ionic loss dealt with in the last paragraph. Therefore, if l is the distance between the electrodes, the subsidiary condition

$$l > \frac{E\frac{e}{m}}{2\pi^2\nu^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

must be fulfilled, if the simple theory given above is to be applicable. Clearly, if l is less than the R.H.S. of (2), the ionization mechanism will approximate to that occurring in the D.C. discharge. This explains why it was stated in Section II. that a typical H.F. discharge takes place at low pressures only when the electrodes are far apart, the distance depending on the frequency. The "typical" H.F. discharge referred to is therefore defined as one in which the amplitude of oscillation of a free electron in the field is less than the inter-electrode distance. This definition is convenient, but, of course, inequality (1) will not define the field necessary to initiate a discharge until the inter-electrode distance is many times the amplitude of the free electron's oscillation.

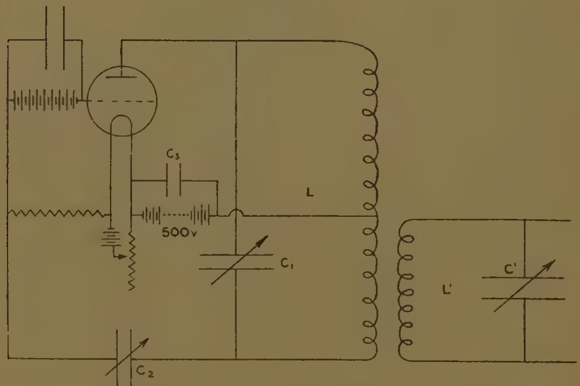
IV. *Preliminary Experiments.*

Having developed the theory of the discharge to the extent described above, the writer next attempted to conceive experiments which would produce evidence for or against it. These took the form of measuring the striking potential (V_s) for a discharge between two approximately plane internal electrodes as a function of the frequency ν and the gas pressure p. In order to conform to condition (2) above it was necessary to have the tube as large as possible, and it was in experimenting with the first of these that the importance of the other conditions laid down in Section II. became evident. The first tube used was a cylinder of length 30 cm. and diameter 6 cm. The diameter of the plate electrodes (slightly convex) was 5 cm., and the inter-electrode distance

was 20 cm. The electrodes were of aluminium in one case and of steel in another, sealed into the glass tube by thick pieces of platinum.

The Gas System.—The necessary vacuum was obtained by means of a three-stage mercury diffusion pump backed by a Hyvac rotary pump. Two pressure gauges were used: one, a Gaede shortened McLeod gauge, reading from 0·1 mm. to 0·0001 mm. and detecting a pressure of 0·00001 mm.; the other, a simple McLeod gauge, reading with an accuracy of 5 per cent. between 3 mm. and 0·1 mm. The gas experimented with was hydrogen, fed by means of a capillary tube from a Hoffmann generator (BaOH as

Fig. 1.



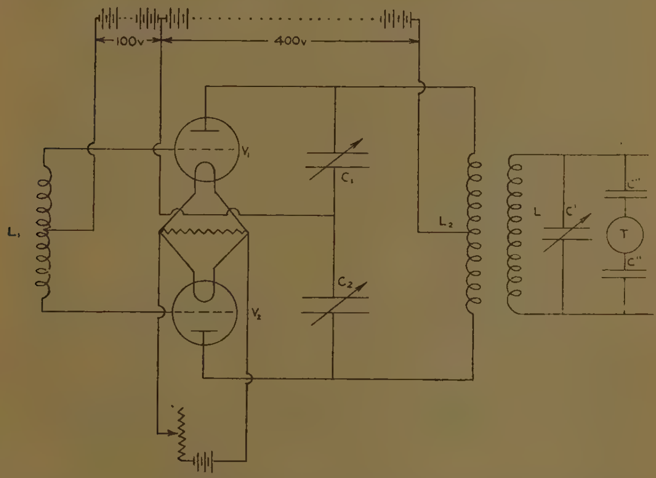
the electrolyte) of special design through tubes of soda lime, calcium chloride, and phosphorus pentoxide. At appropriate points in the system liquid air traps were inserted. All joints were glass-sealed. It was possible to adjust the gas leak through the capillary to maintain a constant flow of gas through the discharge-tube at any pressure between 3 and 0·01 mm. of mercury.

The High-Frequency Oscillators.—Two forms of oscillator were used with success in these experiments. It is necessary that the potential across the discharge-tube should approximate as closely as possible to a sinusoidal wave, and that there should be no appreciable constant potential difference between the electrodes. For frequencies between

10^6 and 10^7/sec. the circuit of fig. 1 was found to satisfy the first of these conditions very well. The use of electromagnetic coupling satisfied the second condition automatically.

Leads were taken from the terminals of the condenser C' to the discharge-tube electrodes. The ratio L'/C' was large in order to produce a large difference of potential. On the other hand, C' was not made as small as it might have been for reasons which will be explained later. The uniform change in the potential across C' was brought about by a fine variation of the filament current. The

Fig. 2.



value of the grid-biassing voltage was adjusted to make the grid current as small as possible. Other precautions were taken (see D. F. Martyn, "Frequency Variations in the Triode Oscillator" [20]) to ensure that the frequency did not vary appreciably with the filament current. The valve used was a Mullard 40-watt transmitter.

For higher frequencies (10^7–10^8/sec.) the circuit shown in fig. 2 was employed.

The matched valves V_1 and V_2 were Marconi L.S. 6A, dissipating about 25 watts each. The oscillating circuit $L_2\,C_1\,C_2$ is tuned to the grid circuit (L_1 and the inter-electrode capacities of the valves). To vary the frequency

it was therefore necessary to change one or both of the coils L_1 and L_2, and to retune the condensers C_1, C_2. This oscillator is remarkably efficient. Moreover, owing to the centre-tapped design, it is free from "capacity effects" when tuning is in progress.

The Measurement of V_s.—The potential V_s was measured by the well-known "heater and thermocouple' method, the instrument T being placed near the potential node between two approximately equal condensers C'' (fig. 2). This method has been extensively employed by Professor Townsend's collaborators. Occasionally it was found advisable to shunt the condensers by a non-inductive resistance with a centre-tapping connected to earth. Many other methods of measuring V_s were tried, but they are all open to serious objections. Even with the heater and thermocouple the absolute precision of the result depends upon the precision with which the capacities can be measured, and since C'' was usually of the order $1 \mu\mu$ F, this was small. As a consequence, the absolute values of V_s given in the sequel may be in error by one part in forty; the relative values are probably correct to within 1 per cent.

The Measurement of the Frequency.—The measurement of the oscillation frequency was carried out with a simple absorption meter (General Radio Co.) of range $2{\cdot}1 . 10^7$–$1{\cdot}2 . 10^6$/sec. The instrument consisted of a standard variable condenser with four fixed inductances. After tests and adjustments had been made the meter was found to have a mean error of less than 1 per cent. It was necessary that the frequency should be known at least as accurately as the capacities C'', since both appear in the same way in the expression for V_s in terms of the current flowing in T.

General.—In these preliminary experiments the pressure of the gas in the discharge-tube was measured by means of the mercury gauges. Liquid air was not used round the traps; it is almost impossible to use the former in such a way that mercury vapour will be excluded from the tube over a long period. From the theory given in Section III. it will be seen that the important potential across the tube is the *peak* potential, and not the R.M.S. The design of the oscillators was such that it could be

safely assumed that the peak factor was $1{\cdot}41 \pm 0{\cdot}01$, but this was tested directly for certain frequencies by means of a cathode-ray oscillograph. In all that follows V^s will be given as a peak potential.

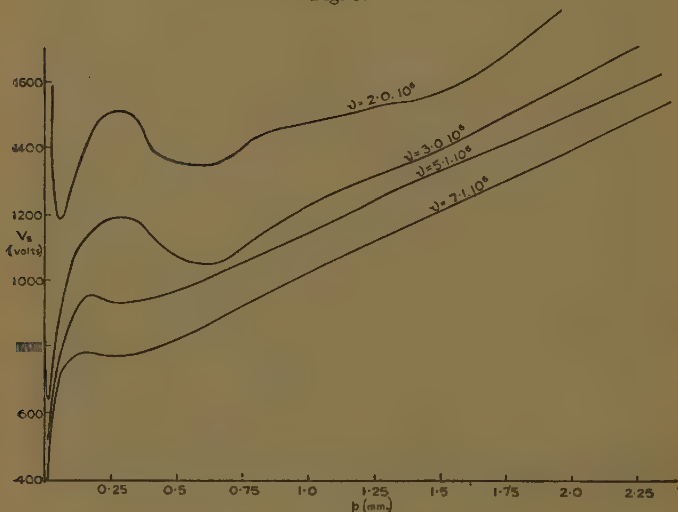
Results.—For the measurement of the striking potential there is only one satisfactory *modus operandi*. The potential across C' is slowly increased, the readings of the thermocouple circuit galvanometer being continually observed. The galvanometer reading *immediately before* the discharge commences is taken. This gives the required potential V_s.

The reason for this procedure is as follows:—When a glow begins, both the capacity and the resistance of the discharge-tube change, so that the circuit $L'C'C''$ is thrown out of resonance with the oscillator. At high gas pressures, where the discharge which takes place at V_s is bright and carries a large current, the detuning of the circuit may be considerable. In fact, apart altogether from the discharge, the variation of the gas pressure in the tube may cause a variation in the tuning if the frequency of the oscillation is very high or the capacities C' are too small. For this reason the capacity C' was kept as large as possible (as mentioned already); it is important that the current flowing in the main oscillator should not vary too much with the load in the coupled circuit. However, no matter how the circuits are adjusted, immediately the discharge begins the reading of the thermocouple circuit galvanometer decreases abruptly, sometimes by an amount corresponding to $\frac{1}{2}V_s$.

At this point an interesting property of the discharge may be described, a property which is characteristic of both pure and impure gas. If a discharge with internal electrodes is initiated at the relatively high gas pressure of 1 mm. of mercury, it begins usually as (1) a symmetrical glow, both the electrodes exhibiting the usual D.C. cathodic phenomena, and the middle of the tube containing a typical anode glow with (in all but the widest tubes) symmetrical striations. If the potential across the tube is now decreased slowly, this régime changes at one point abruptly to another, where (2) the striated column is longer and the cathodic glows become convex towards the electrodes. As the potential is still further reduced,

this again changes abruptly to (3) a faint egg-shaped luminescence near the centre of the tube. This last form is the well-known plasmoidal glow. Thus at high pressures there are three typical régimes, and the discharge begins with (1). At lower pressures the discharge begins at V_s with (2), but can be made to exhibit (1) by increasing the potential. At still lower pressures the discharge begins at V_s with (3), and usually can be made to exhibit (1) by *greatly* increasing the power available.

Fig. 3.



Régime (2) is in abeyance at the lowest pressures. It must be emphasized that these effects have been observed in gas which was entirely free from mercury vapour as well as in impure gas.

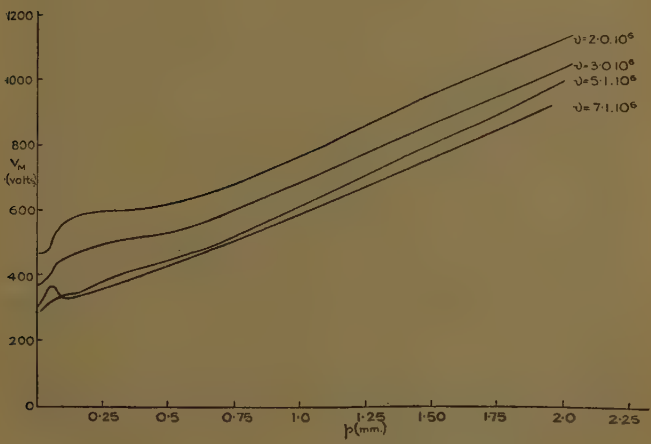
The quantitative results of these preliminary experiments are collected in fig. 3 in the form of graphs showing the variation of V_s with the gas pressure p for four values of the frequency ν.

At the same time and for comparison the "minimum maintenance potential" V_m was measured as a function of p and ν. In this case circuit L′C′C″ was tuned while the discharge was running. This procedure is not so definite

as with V_s, for the tuning varied continuously with the current flowing in the tube. The results are shown in fig. 4 as four graphs of V_m against p for different frequencies.

The Current-Voltage Characteristic of the Discharge.—From these experiments also certain conclusions were reached with regard to the general shape of the current-voltage characteristic of the discharge. Some writers, notably Kirchner (8) and Brasefield (14), have stated that they found it impossible to measure V_s accurately—that they could not obtain consistent results. For that reason

Fig. 4.



they measured either V_m or the potential necessary to produce a given current through the discharge. Now, in order that any discharge may begin, there must be at least one electron in the gas, and under these circumstances *some* potential V_s may be found at which ionization will begin. But, as has been shown by Penning (21), Taylor (22), and others, the starting potential is actually a function of the number of ions present between the electrodes, where, of course, that number is very large. In the D.C. discharge (for which this has been demonstrated) it is almost impossible to arrange matters so that a large number of ions is present in the gas before a discharge takes place, for the constant unidirectional electric field

sweeps them away as fast as they are created. Hence, unless a special technique is employed, no matter how soon after the suppression of a D.C. glow the voltage is again increased to its sparking value, the ions which were present have almost entirely disappeared, and the discharge does not begin until the static sparking potential has been reached. A noteworthy exception to this is the oscillatory discharge of a condenser [23]. In this case the time between two successive potential peaks is short, and in the interval between them the electric field has changed its direction, so that once the discharge has been struck, it recommences in succeeding oscillations at approximately its minimum maintenance potential. But clearly, although the oscillatory discharge is brought about by a constant potential difference, it approximates in its mechanism to the H.F. discharge, so that the case hardly can be called exceptional.

In the case of an H.F. electric field after a discharge has been suppressed the ions may remain in the gas for many seconds, escaping neutralization at an electrode or wall. Hence, when the voltage is again increased, there is what may be termed a potential source of current present, and, if the H.F. voltage-ampere characteristic is similar in shape to the D.C. characteristic, the discharge will start at a lower voltage than the static sparking potential. The writer has actually been able to start a discharge at a voltage only slightly higher than V_m by rapidly increasing the potential across the tube immediately after the extinction of the previous glow, demonstrating the existence of the effect. It is probable that Kirchner's difficulty in securing consistent results for V_s is to be traced to this cause. The effect becomes more and more troublesome as the frequency is increased. Brasefield's case is complicated by the fact that negative charges would form on the walls of his electrodeless tube. In the present experiments the results for V_s were found to be quite consistent and could be repeated at any time. This was due presumably to the constant flow of gas through the tube, and to the fact that several minutes were allowed to elapse between the readings.

By placing a heater and thermocouple between two exactly similar discharge-tubes and applying the H.F. potential across the two tubes in series, the writer has been able to investigate directly the current-voltage

characteristic between certain narrow limits. For two tubes identical with that described above it was found that between 5 and 20 milliamperes the conduction current increased linearly with the voltage. This corresponds to the part of the D.C. characteristic in which the same currents are obtained. For lower values of the conduction current the capacity component was of the same order of magnitude. This made it impossible to investigate the really important part of the curve. Fortunately, further evidence was indirectly obtained in an interesting experiment which will now be described.

As has been mentioned, the mode of controlling the potential across the tube was to vary the filament current in the valve or valves. This actually controls the power output of the oscillator and therefore the power available in the discharge-tube circuit L′C′. The potential across C′ for a given value of the filament current depends upon the current flowing in the discharge—the greater the load on the circuit the smaller the voltage across C′. The glow was started and the filament current slowly reduced, readings being taken of the potential across C′ in the usual way. When the glow was very weak, it was observed that decreasing the filament current *increased* the potential across C′, and this continued until the discharge went out altogether. It was verified that the effect was not due to a change in the natural frequency of the circuit L′C′, so that the inference is that for small currents through the discharge a decrease in the energy input causes a decrease in the conduction current through the tube and an increase in the voltage across the tube. It would therefore appear that the small-current part of the volt-ampere characteristic is negative, as it is in the case of the D.C. glow.

V. *A Test Experiment.*

The curves of figs. 3 and 4 were obtained from observations of V_s and V_m at pressures between 0·01 mm. and 2·5 mm. The pressure of mercury vapour (saturated) at room-temperature is 0·0007 mm., so that, since liquid air was not continuously employed round the traps, the percentage impurity in the gas varied approximately between 7 and 0·03 per cent. To begin with, it was assumed that the presence of this impurity would not disturb the results seriously; that this assumption was

completely false was shown by a later test experiment which will now be described.

A new discharge-tube, similar to those used previously, but free from mercury contamination, was connected into the system with a liquid air trap between it and the pumps and gauges. From the moment of scaling the tube into position liquid air was kept round all the traps in the system. Then a set of readings of V_s at different gas pressures was rapidly taken, as soon as it could be safely assumed that the discharge-tube was well outgassed. The frequency of the applied potential oscillation was $5 \cdot 10^6$/sec. It was found that the new curve $V=f(p)$ was quite different from the corresponding one of fig. 3. The latter has a minimum followed by a maximum, but in the former V_s steadily decreased with decreasing pressure to a very sharp single minimum at $p=0 \cdot 1$ mm. As the pressure was still further diminished, V_s increased very rapidly, becoming too large to be measured at $p=0 \cdot 02$ mm. The hump of fig. 3 and the diminution of V_s at low pressures entirely disappeared in the mercury-free tube.

The test experiment clearly demonstrated the futility of attempting to use a tube or gas which was not very free indeed from impurity. The true mode of variation of V_s with the pressure may be completely masked by the presence of mercury vapour. This observation is in accord with the experience of other workers, notably Townsend and Donaldson(13): the writer has offered a tentative explanation of the phenomenon in a previous paper(24). Reference may also be made to some interesting observations of Headrick and Duffendack(25).

VI. *Quantitative Experiments in Mercury-Free Tubes.*

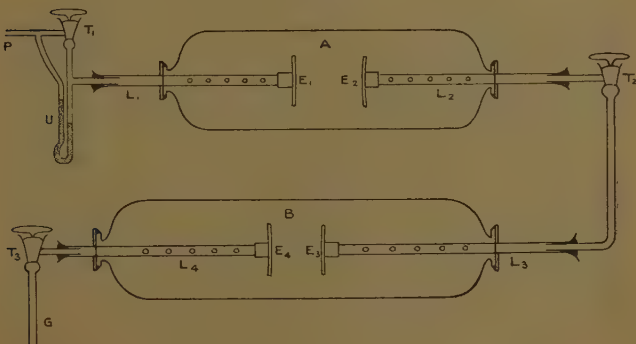
The Improved Apparatus.—In order to eliminate impurities and at the same time improve the precision of the observations it was decided to employ an entirely new system of pumps, gauges, and discharge-tubes. From the outset mercury vapour was rigorously excluded from the apparatus. The pumping installation (supplied by Metropolitan-Vickers) consisted of the now well-known oil diffusion pumps backed by a single-stage rotary pump. These were connected at P to the remainder of the system shown in fig. 5.

The dimensions of the discharge-tubes A and B were as follows :—

	Length (cm.).	Diameter (cm.).
Tube A	50	15
Tube B	60	15

The copper tubes $L_1L_2L_3L_4$ acted in two capacities. Each was 1·25 cm. in outside diameter and pierced with many large holes along that part of its length which lay inside the vessel A or B. Hence they acted as connecting tubes, joining A to B, A to the pumps, and B to the gas purification system. But the tubes also acted as leads to

Fig. 5.



the electrodes $E_1E_2E_3E_4$, each of which consisted of a copper disk of diameter 7·5 cm. made slightly convex on its outside surface. These electrodes were screwed on to the ends of the copper tubes. Referring to fig. 5, the glass tube from the pumps at P was connected through the tap T_1 and the manometer U with the tube L_1. The manometer contained Apiezon oil B as the liquid. The tap T_2 enabled container A to be isolated from B, while tap T_3 controlled the flow of gas into the system. Both discharge-tubes contained similar electrodes, but, while those of B were fixed at a distance from each other of 7·5 cm., one of the electrodes of tube A was movable, being held in a well-fitting copper sleeve sealed with Q compound. The distance between the electrodes E_1 and E_2 could be read directly from a scale engraved on a movable

copper tube L_2 outside of A. For the experiments to be described in the present communication the distance E_1E_2 was kept constant at 10 cm. The apparatus shown in fig. 5 was connected to the gas purification system and Hoffmann generator described in Section III. by a long fine capillary tube. All joints were made with Apiezon wax W, and the taps were sealed with Apiezon oil B, using grease L as the lubricant.

The potential V_s and the frequency ν were measured by the methods already described; the oscillator used was that shown in fig. 2. The outstanding problem was the measurement of the gas pressure, since mercury had been banished from the system. This difficulty was overcome in the following manner. The tubes A and B were filled with the gas (hydrogen) to a pressure of approximately 6 mm. of mercury, and this pressure (p_0) was read directly on the oil gauge U by means of a microscope. (The specific gravity of the oil was measured and the makers' figure (0·871) verified.) Then tap T_2 was closed so that A was isolated from B, tap T_1 was opened, and A was completely evacuated. Tap T_1 was closed and Tap T_2 was reopened, sharing the remaining gas between the two discharge-tubes. The new gas pressure p_1 was then observed. The cycle of operations was repeated, giving p_2. If the cycle is performed n times, by Boyle's law the final pressure p_n is given by

$$p_n=\left(\frac{V_B}{V_A+V_B}\right)^n p_0=k^n p_0, \quad . \quad . \quad . \quad (3)$$

where V_A and V_B are the volumes of the containers A and B respectively. From the observations taken k was deduced. Four sets of readings were taken, the pressures p_0, p_1, p_2, p_3, being measured in each case. From these a mean value of k was calculated. It was 0·582. The microscope used to estimate the heights of the oil columns read to 0·05 mm., and the height corresponding to p_0 was approximately 90 mm., so that the pressure was observed with considerable accuracy. All the values of k deduced from the observations lay within 1 per cent. of each other. The mean value is probably correct to $\frac{1}{4}$ per cent. This value of k was checked at various times during the subsequent experiments, and in no case was any appreciable error discovered. As a precautionary measure the volumes of the tubes A and B were estimated by filling them with

water before they were assembled. This preliminary experiment (inexact, since the connecting tubes could not be properly accounted for) gave $k=0{\cdot}581$. The volume of tube A and its connexions was estimated to be 6·67 litres.

When readings of V_s were being taken, only the initial value of the pressure was read directly on the manometer. After the value of V_s corresponding to this pressure had been observed, the cycle of operations giving $p_1=kp_0$ was carried through, and p_1 served as the new gas pressure at which V_s was observed. In the same way $p_2, p_3, p_4, \ldots$ were obtained. Hence after p_0 had been read directly, the subsequent pressures were deduced by means of equation (3), and pressures too small to be read accurately on the manometer were determined with precision in this manner. Incidentally, the logarithmic variation of p is exactly what is required to obtain a useful V_s-p graph.

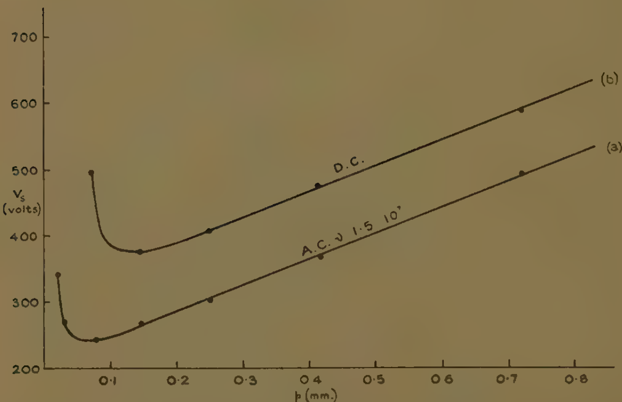
To use the above method of estimating low pressures it is necessary to be sure that the tubes contain no condensible vapour and are thoroughly outgassed. Before the present experiments were begun, the tubes were pumped out, using both diffusion pumps, for about 100 hours in three- to four-hour periods. In each second interval between the pumping periods the tubes were washed out with hydrogen. At the end of the hundred hours pumping tube B, which was furthest from the pumps, was so hard that after being cut off from the pumps for twelve hours it showed no discharge when a potential of 40,000 volts was applied across its electrodes. It was assumed, therefore, that the system was tolerably well outgassed. Such a test indicates the extremely low vapour pressures of the Apiezon products used. Clean dry hydrogen was then admitted to the apparatus and experiments were begun.

Experiment I.—The first experiment consisted in measuring V_s at different gas pressures for three frequencies greater than 10^6/sec. A typical result is shown in curve *a* of fig. 6.

Experiment II.—This consisted in measuring V_s at different gas pressures, using a steady potential between the electrodes. The constant potential was obtained by means of a high-tension generator connected across a potentiometer fitted with a fine control. The result is shown in fig. 6, curve *b*.

Experiment III.—A low-frequency valve oscillator was constructed, and the primary of a small induction coil was included in the oscillatory circuit. High alternating potentials of frequency 850 ± 10 cycles per sec. were produced across the secondary of the coil, the terminals of which were connected to the electrodes of the discharge-tube and to an electrostatic voltmeter. The aim of the experiment was to measure V_s at the relatively low frequency of 850/sec. However, as was indicated in the previous section, it is the peak potential between the electrodes which is important from the point of view of the

Fig. 6.



theory of the discharge, and the electrostatic voltmeter (a Kelvin, Bottomley, and Baird Standard Instrument) read not the peak, but the R.M.S. voltage. It was therefore necessary to ascertain the peak ratio, for it was suspected that the induction coil was not acting as a distortionless transformer. Experiments indicated that with a capacity of 0·01 μF. across the secondary, the peak factor was sufficiently constant over the voltage range to allow of extrapolation from a graph of the peak voltage against the R.M.S. voltage, and the coil was therefore used in that manner. The striking potential V_s was then measured at different gas pressures as before. The result will be given later.

It was soon discovered that small unexplained variations in the nature of the samples of gas used produced small changes in the value of V_s for any given pressure and frequency. For a given sample of gas at a given pressure the value of V_s at a given frequency would remain constant for days, but each new sample of gas was characterized by a change (of the order of 5 per cent.) in this value. Hence many sets of experiments were made, and the results were averaged. Ultimately, when a set of readings at frequency ν was being taken, simultaneous readings of the D.C. value of V_s were also taken. These acted as a check upon the nature of the gas.

Conclusions.—The experiments demonstrated the following facts :—

(*a*) Over a considerable part of the range of gas pressures used V_s is a linear function of p.

(*b*) The gradient $\frac{dV_s}{dp}$ in this region is independent of the frequency. The gradient is also the same for a D.C. discharge. The mean value of the gradient as found from the present experiments is 390 volts per mm. of mercury, the distance between the electrodes being 10 cm.

(*c*) If the straight line $V_s = A + Bp$ is produced to cut the V_s-axis, the value A of V is a function of the frequency.

These facts may be summarized in the following table :—

TABLE I.

A	307	325	230	207	160
$\frac{1}{10}$B	38·8	39	39	39	39
ν	0	851	$8{\cdot}1 . 10^6$	$1{\cdot}5 . 10^7$	$2{\cdot}0 . 10^7$

A and $\frac{1}{10}$ B are in volts, ν is in cycles/sec.

Discussion of the Results.—It is now well established that the static sparking potential V is (for a given distance between the electrodes and a given frequency) a linear function of the gas pressure, provided that the

latter is definitely greater than the critical pressure. In algebraic form

$$V_s = A + Bp. \qquad (4)$$

This law is only true for pure gases contained between electrodes so arranged that the electrostatic field due to their charges is approximately uniform. For D.C. discharges this law may be deduced from Carr's investigations [26], while for H.F. discharges the results obtained by Rohde[27] and in the present experiments verify the law excellently. Carr's results for hydrogen between parallel plates 1 cm. apart may be expressed by

$$V_s = 280 + 38{\cdot}2\,p \qquad (5)$$

where V_s is in volts, and p in mm. of mercury.

It is also well known that V_s is (for a given gas pressure and frequency) a linear function of the distance d between the electrodes, again provided that the latter is greater than the critical value, and that the electric field is uniform. In algebraic form

$$V_s = C + Dd. \qquad (6)$$

This result is also in agreement with the best modern investigations. The writer has also verified equation (6) by using the two discharge-tubes A and B simultaneously, the electrodes in B being 7·5 cm. apart. It would appear, therefore, that Paschen's Law may be extended to the high-frequency discharge, and we may write

$$V_s = a + bpd, \qquad (7)$$

where a and b are constants for a given pair of electrodes in a given gas and for a given frequency of the applied oscillation. From the present experiments in hydrogen

$$V = f(\nu) + 39pd, \qquad (8)$$

where p is in mm. of mercury and d is in cm. This may be compared with Carr's result given in equation (5).

The Theory of the D.C. Discharge.—It is almost universally admitted that in order to produce a discharge at a very high frequency it is not necessary that there should be any ionizing action in the gas other than that due to electron impact. On the other hand, it is well known that ionization by the impact of negative ions is insufficient in itself to produce a self-sustained discharge

at low frequencies. It would seem, therefore, that the considerable differences which have been shown to exist between the H.F. initiation potentials and the D.C. initiation potentials must be due, at least in part, to the fact that the extra ionizing factor is unnecessary in the former case. There is still some dubiety, however, with regard to the nature of this extra ionizing action. According to some[28], it is ionization by the collision of positive ions with molecules of the gas—a process exactly analogous to the ionization by collision of negative ions, but requiring on the average a considerably greater energy dissipation per collision to produce a pair of ions. According to others [29], the extra ionizing action is located at the cathode, being due in some manner to the absorption by the latter of the positive ions. Such a process is not in the least analogous to the ionization by negative ions.

But the present experiments have shown that b of equation (7) is independent of the frequency within the range 0–$2 \cdot 10^7$ cycles per second, and clearly b is the quantity which is determined by ionization processes in the gas due to ionic collisions. In fact, b is the true X/p of Townsend's theory. If, therefore, b does not vary with the frequency, even when that is high enough to cause the minimum value of V_s^* to be only 1/2 its D.C. value, it seems likely that it does not include any factor depending upon ionization by collision of positive ions. On the other hand, the variation of a with the frequency (Table I.) is what might be expected on any of the alternative theories.

The present experiments are not, however, conclusive upon this point. It is possible that the highest frequency used is not high enough. The experiments will require to be continued to a frequency of 10^8 cycles per sec. This could not be done with the discharge-tubes which have been described here, as the electrode leads are too long to make accurate measurements of V_s possible.

Comparison of the Experiments with the Electrostatic Theory.—For gas pressures decidedly greater than the critical value inequality (1) becomes

$$E > \frac{V}{K} p. \qquad (9)$$

Clearly this does not agree with any of the results obtained. Even when a constant is added to the R.H.S. to represent $f(\nu)$ in equation (8), this constant cannot be used in inequality (1) to represent the results at pressures equal to or lower than the critical value. The fact is that the frequencies used are still too low to make useful comparison possible. The fact that $\frac{dV_s}{dp}$ is independent of the frequency has the same limited significance here as in the last paragraph. So far as the experiments go, there is nothing to indicate that the theory given in Section III. does not hold for suitable values of the frequency. If it can be shown that the theory *is* substantially correct, then the constant b (39 volts per cm. per mm. pressure) will acquire a new significance.

General Conclusion.—So far as verification of the theory is concerned, the present experiments demonstrate that it is necessary to use very high frequencies indeed. To do this it is essential that the leads from the electrodes to the voltmeter measuring V_s (no matter what form that voltmeter takes), must be short compared with $\frac{1}{4}\lambda$, where λ is the wave-length of the oscillation. A moment's consideration will show that to do this, and at the same time conform to the other conditions laid down in Section III., it is necessary to use small distances between the electrodes. Since the amplitude of oscillation of a free electron varies as $1/\nu^2$, this is possible. It would, however, be very much more convenient if the use of external electrodes was permitted.

I wish to thank Professor E. Taylor Jones and the Court of the University of Glasgow for the facilities and apparatus for the experiments. To Mr. William Reid I am very much indebted for his assistance in mechanical matters and for his careful construction of the oscillators.

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(28) Townsend, 'Electricity in Gases,' chap. ix. (1915).
(29) Holst and Oosterhuis, Phil. Mag. xlvi. p. 1117 (1923).
Taylor, Proc. Roy. Soc. A, cxvii. p. 508 (1928), and previous papers.

June, 1934.

LXVIII. *The Diurnal Variation of the Space Charge and its Effect upon the Potential-Gradient.* *By* C. H. Dwight, *Department of Physics, University of Cincinnati* *.

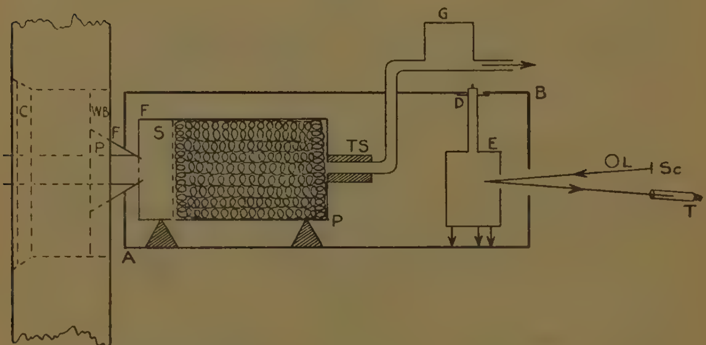
Introduction.

THE measurement of the strength of the earth's electric field, or "potential-gradient," involves the placing, at a known distance from the ground, of a carefully insulated conductor, carrying an ionizing agent (such as minute quantity of a radioactive material) which will form ions in the neighbourhood and tend to let the conductor take up the air potential in the immediate region. The conductor is connected to an electrometer, so that the potential difference between the air and ground can be measured. The potential - gradient varies with the environment, meteorological conditions, the hour of the day, and the time of year. The atmosphere of the earth contains ions, positive and negative, of various sizes: electrons, ionized atoms, molecules, groups of molecules, microscopic dust, fine particles of water, and soot. These

* Communicated by the Author.

bodies have been brought into existence as electrical entities by the action of ultra-violet light from the sun, cosmic radiation, friction of dust, formation of fine spray, radioactivity of the air and soil, and combustion. Their motion is due to two causes: the mechanical action of the air convection currents, and the electric field, represented by the potential-gradient. Their motion constitutes an electric current from the upper air to the earth, the value of this current at a given region being due to the number of ions in unit volume, their velocity under a unit gradient, and their charge. From mechanical considerations the size of the particles will largely determine the velocity. The net quantity of electricity (sum of positive and

Fig. 1.



negative) per unit volume of air is called the "space charge," and its variation is controlled largely by the same factors that control the potential-gradient. That has been investigated by Brown *.

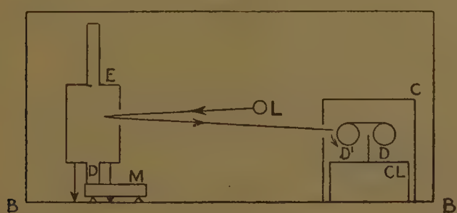
Apparatus Used.

The space charge was measured after the manner suggested by Obolensky †. The modified apparatus was as shown in fig. 1. FP was a Faraday pail, approximately 10 cm. in diameter and 22 cm. in length, filled almost full with closely-packed steel-wool. The latter was held

* Brown, Terr. Mag. v. (35) p. 1 (1930); v. (38) p. 161 (1933).
† Obolensky, *Ann. d. Phys.* 4th ser. v. (77) p. 644 (1925).

in place by a perforated disk S. The pail was connected to the gas-meter G and the aspirator (not shown) by means of a sulphur tube TS, and was connected electrically to one pair of quadrants of the electrometer E. The pail was supported by sulphur pyramids. The entire assembly was placed in the foil-lined wooden box AB, with the foil lining grounded. A funnel F led into the box from the window-board WB, so that the air aspirated from outdoors was directed into the pail. A metal tube P extended to the outside of the window, the entire area of which was covered with a grounded wire-netting C. A small electric lamp L, scale Sc, and telescope T enabled deflexions of the electrometer to be read. The capacity of the system was found and a calibration curve prepared, so that deflexions could be interpreted as ESU of quantity when the ions

Fig. 2.



were drawn into the steel-wool and imparted their charge to the electrometer and pail system. The sensitivity of the electrometer was of the order of 2000 mm. per volt at the standard scale distance.

The air potential was measured with a low-sensitivity electrometer E (fig. 2), which was provided with a copper damping vane D and a damping magnet M. Light from a small lamp L was reflected from the electrometer to a pair of drums DD′, turned by clockwork CL in a cabinet C. The drums carried bromide paper, upon which the motion of the spot of light was photographically retained. The needle of the electrometer was connected, through the lead-in (fig. 3), to the collector-wire on the outside of the building. The wire was supported horizontally at a distance of about 1 metre from the side of the building and at a height of 10·3 m. from the ground. The wire had

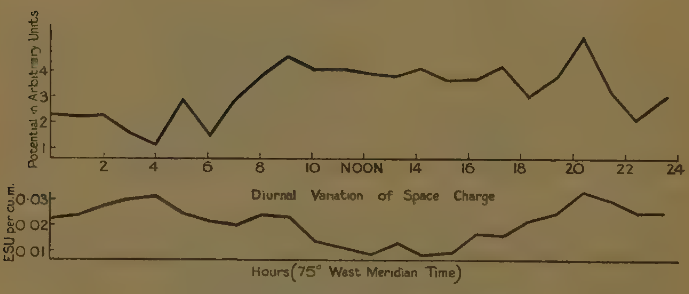
a length of 14·5 m. Sulphur insulation was used throughout. One pair of electrometer quadrants was connected to ground, the other pair to the positive terminal of a 45-volt battery, the negative terminal of which was grounded.

Experimental Results.

Table I. gives the hourly means for the electrometer deflexions which represent air potentials. One unit on the vertical axis corresponds to a potential difference

Fig. 3.

Fig. 4.



Cincinnati (U.S.A.). Spring of 1930.
Diurnal Variation of Air Potential.

of about 20 volts. Table II. gives the hourly means of the space charge in ESU per cubic metre. While the two elements were not measured always simultaneously, they were determined on days which were characterized by the same weather conditions. Readings were taken every fifteen minutes, averaged, and plotted as values of the elements for the mid-point of the hour period. It will be observed from the graphs that a maximum in the value of the air potential is approximately coincident with a minimum in the value of the space charge. It is possible that the air potential depends upon the latter.

TABLE I.—Variation of Air Potential. (Arbitrary units.)
75° west meridian time.

Hour	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
Mean value	2·3	2·2	2·2	1·6	1·2	2·8	1·4	2·9	3·8	4·5	4·0	4·0	3·8
No. of observations	15	10	16	14	10	7	10	14	27	43	48	57	51
Hour	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	24.	
Mean value	3·7	4·0	3·6	3·6	4·1	2·9	3·6	5·2	3·1	2·0	2·8	—	
No. of observations	73	63	48	41	34	21	21	21	18	22	17	—	

TABLE II.—Variation of Space Charge. (ESU per cu. m.)
75° west meridian time.

Hour	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
Mean value	·022	·023	·027	·030	·031	·025	·022	·020	·024	·023	·014	·012	·009
No. of observations	31	32	31	26	24	28	32	38	54	79	60	80	41
Hour	13.	14.	15.	16.	17.	18.	19.	20.	21.	22.	23.	24.	
Mean value	·013	·089	·010	·017	·016	·022	·024	·032	·029	·024	·024	—	
No. of observations	75	79	79	78	62	45	57	57	47	39	35	—	

The situation of the apparatus was unfortunate, in that there were trees, bushes, buildings, and a roadway in the vicinity. It was found impracticable to correlate the variations of the atmospheric-electric elements with the changes in the relative humidity, air temperature, barometric pressure, and wind velocity. That is a problem which requires further investigation.

LXIX. *Notices respecting New Books.*

The Atom. By JOHN TUTIN, D.Sc. With an Introduction by Prof. F. SODDY, F.R.S. [Pp. 109.] (London: Longmans, Green & Co., Ltd., 1934. Price 6*s*. net.)

DR. TUTIN has written a book on the atom which is commended to the scientific world by Prof. Soddy, and which, it is asserted on the jacket, " the general reader fond of science will find . . . attractive as well as intelligible."

An alternative atom is proposed. It is pointed out that the scattering experiments of Rutherford do not necessitate the conclusion that the nucleus of the atom is positively charged, or that the mass of the atom is concentrated in it. The observations are equally compatible with the view that the nucleus consists of electrons, and is itself very light, provided that, as is possible with quantum laws, the nuclear electrons are firmly bound to the outer atom. The consequences of this view are developed. The atomic number is simply the number of electrons in the nucleus. The outer structure consists of protons and electrons, either singly or in groups of six types containing two, three, or four protons and a smaller number of electrons. With these bricks the game of atom building is played. It is maintained that with the alternative atom many facts can be explained for which the orthodox atom fails, and in such a way as not to involve a breach in the law of causation.

The author states that " the alternative atom appears to defy all attempts to prove that it is wrong." The essential question, however, surely is whether the structures proposed are compatible with any self consistent scheme, presumably of a quantum-mechanical type, which could be formulated. No attempt at such a formulation is made. The whole development is qualitative, new assumptions and postulates being introduced at will, without any irksome check such as is imposed by the demands of quantitative self-consistency. That a thorough-going reformulation would be necessary can hardly be doubted, for the alternative atom is not a mere

"mirror image" of the orthodox atom. The extranuclear structure consists not of particles of one kind, but of electrons, protons, and groups.

It is the groups, other than the α-particles, which provide the valency linkages. These groups, giving "electro-magnetic resultants," endow the atoms "with both the urge to unite and the means of doing so." The single ("free") electrons in the atoms are supposed to determine the metallic characteristics, and not directly the valency. Thus carbon, oxygen, silicon, sulphur, and chlorine have no free electrons; other atoms, including the inert gases, have one or more.

Great stress is laid on the peculiarities of radioactive disintegration. It is maintained that a purely causal interpretation of the half-period values can be given by linking them up with the characteristics of the perturbed α-particle orbits round the nucleus.

It will be sufficient to give a few quotations to indicate the general character of the argument on other points, as critical of the orthodox view: "It must explain why some atoms emit light and others do not, why some are electric conductors and others insulators, why some are magnetic and others non-magnetic .... It will not explain these simple matters"; "The accepted interpretation of chemical inertia depends on the supposed 'stability' of certain numbers"; "Even the important Geiger-Nuttall formula has not yet received a convincing mathematical interpretation."

With the alternative atom explanations of some of the most baffling phenomena can be found. It will be sufficient to quote how the problem of ferromagnetism is cleared up: "The explanation of ferromagnetism afforded by the new theory is that the iron atom, instead of having its valency links radiating equally in all directions like manganese has three at one 'pole' and three at the other."

The books purports to be a criticism and a constructive suggestion. Sound criticism is to be welcomed; but the first requirement is that the critic should have a clear comprehension of what he criticizes. Here the criticism is largely superficial and based on misapprehensions. The suggestion that the nucleus might be negatively charged is defensible; but the scheme built up is so full of arbitrary features and nebulous speculation as to be practically devoid of value. It might seem that so long a review is uncalled for. On the one hand, however, the book has appeared unduly weighted by implications of its importance, and some counterpoise seems desirable; on the other the book is, of course, not devoid of points of interest. It will at least stimulate scientific readers to consider by what criteria they would judge whether a hypothesis is useful or not.

The Development of Physical Thought. By LEONARD B. LOEB and ARTHUR S. ADAMS. [Pp. xvi+648.] (John Wiley & Sons, New York; Chapman & Hall, London, 1933. Price 23*s*.)

THE authors express the hope, at the end of the preface, that " this book will serve as an inspiring guide to the marvels of modern physics," at which many readers may fear the worst. Their fears are unjustified. The great value of the book lies in the fact that " modern physics " (which occupies about a quarter of the space) is presented in due perspective in relation to the whole historical development; the high lights are not confined to the twentieth century.

After tracing the development of physical science from the earliest times up to the seventeenth century against the wider historical background, mechanics, heat, electricity and magnetism, light, and the new physics are treated in turn. Coherence is given by the general historical approach, and in the individual chapters by the logical development of leading themes. No misguided attempt has been made to avoid a mathematical presentation where this is the natural mode of expression, but the mathematics is such that the arguments should be clear to those whose mathematical equipment is of the slenderest.

The book is an expanded form of survey courses of lectures intended for both arts and science students. For the average arts student the book might be considered difficult; but any one who really wishes to know something about physics, rather than to gather hazy notions about modern marvels, will find it at least as straightforward and as full of human interest as, say, the average book on economic history. The difficulties of physical science as a subject to be read about (apart, that is, from the difficulties of experimental and mathematical technique) are grossly exaggerated.

For the student of physics, whether at an introductory or more advanced stage, the book seems ideal. There is a tendency in introductory courses to excessive preoccupation with the sign conventions of geometrical optics and similar academicisms whose importance, whether from the point of view of insight into physical science or of practical application, is almost negligible. This book provides an appropriate counterblast, and should be of great service to those concerned with the giving of such courses. At many universities in this country courses are given in which due emphasis is placed on the historical and general side as well as on the discipline of formal exercises. This book gives explicit recognition of the desirability of such courses and serves as an admirable model.

It is almost inevitable in so comprehensive a book that a few misleading statements should creep in, as in connexion with the permeability of paramagnetics (p. 409) and the gyromagnetic effect (p. 415). The number of such errors however, appears to be very small.

The book as a whole can be strongly recommended to all who wish to gain a sound and thorough appreciation of the development and achievements of physical science.

Thermochimie. By W. SWIETOSLAWSKI. Translated into French by M. THON. [Pp. xix+379.] (Paris : Librairie Felix Alcan, 1933. Price 60 fr.)

THE author of this work has achieved wide recognition as a thermochemical expert by his important and extensive experimental contributions to the subject. Since the period of active work which is familiarly associated with the names of Marcellin Berthelot and Julius Thomsen the advances in thermochemistry have not been such as to attract widespread attention at any particular moment, but steady and continuous progress has produced a net addition to our knowledge of this branch of chemistry which is of great significance. The present position is admirably summarized in the author's volume.

The contents are divided into four sections, the first of which deals primarily with methods of calorimetric measurement. It gives a succinct account of the distinguishing features of ordinary, adiabatic and isothermal calorimetry, and the technique of microcalorimetry is also described. Particular attention is devoted to the various matters which are involved in bomb calorimetric work of the highest possible degree of precision. The second part is concerned with the analysis of numerical data, with especial reference to the well-known principle of additivity. This section contains much new material relative to the thermochemistry of a wide variety of groups of organic compounds.

In the third part consideration is given to the thermal data for simple chemical compounds in relation to the atomic linkages which have attained general recognition as a result of recent developments of the theory of the electronic structure of such compounds.

Under the title of chemical affinity, the fourth section provides an account of the relations which are the outcome of the application of thermodynamical principles.

The exposition is singularly clear, and the book as a whole represents a bibliographical contribution to physico-chemical literature which should be of the greatest value to workers in many modern fields of research.

American Mathematical Society Colloquium Publications.—Vol. XVI. *Algebraic Functions.* By GILBERT AMES BLISS, Professor of Mathematics in the University of Chicago. (American Mathematical Society.)

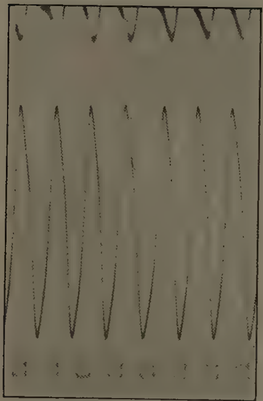
THE theory of Algebraic Functions has suffered considerably in the past from the painful alternatives of intelligible but logically incomplete presentations in terms of Algebraic Curves on the one hand and rigorous but very laborious treatment in terms of Riemann Surfaces on the other. This book of 216 pages is a further presentation in the latter form, and the first six chapters, consisting of the usual work leading up to Abel's Theorem, form an excellent course for the beginner in the subject, whether he be analyst or geometer. Topological considerations have not been allowed to obscure the general sense, nor has the analysis been allowed to become over burdensome. It may, perhaps, be wondered whether anything other than inspired teaching can overcome the difficulties which the purely geometrical mind finds in approaching a theory demanding rigorous complex variable analysis; the writer of this book has certainly made great efforts in this direction. The closing chapters of the book are devoted to Birational Transformations and the Reduction of Singularities by Transformations.

EDITORIAL NOTE.

IN the Phil. Mag. xvi. p. 164 (1933) Mr. K. Sezawa criticized a paper by Prof. Dr. Eng. E. Schwerin concerning the vibrations of turbine blades. Prof. Schwerin now writes to point out that there is an essential difference between the problem treated by Mr. Sezawa and his own. While Mr. Sezawa determines the frequency of a group of blades, each of which vibrates in exactly the same manner (with respect to amplitude and phase), Prof. Schwerin investigated the frequency of a group, the end-blades of which vibrate in a different manner from the middle blades. For solving such a problem the use of simultaneous difference equations is necessary, and it is not possible to find the solution by considering only one blade and its piece of shrouding as Mr. Sezawa does.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

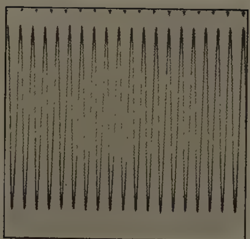
FIG. 4.



(a).

(b).

FIG. 5.

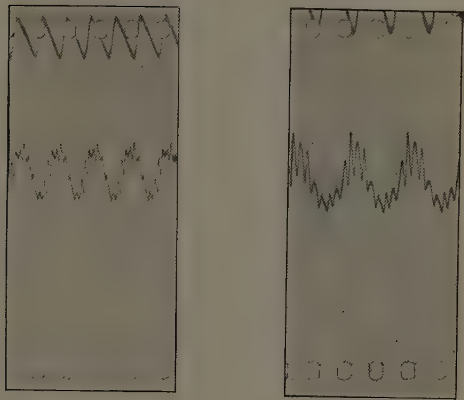


(a).

(b).

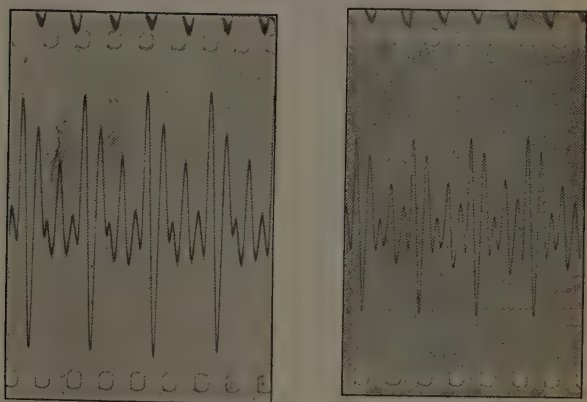
Oscillograms of alternating currents recorded at the far end of the line.

FIG. 6.



(*a*). (*b*).

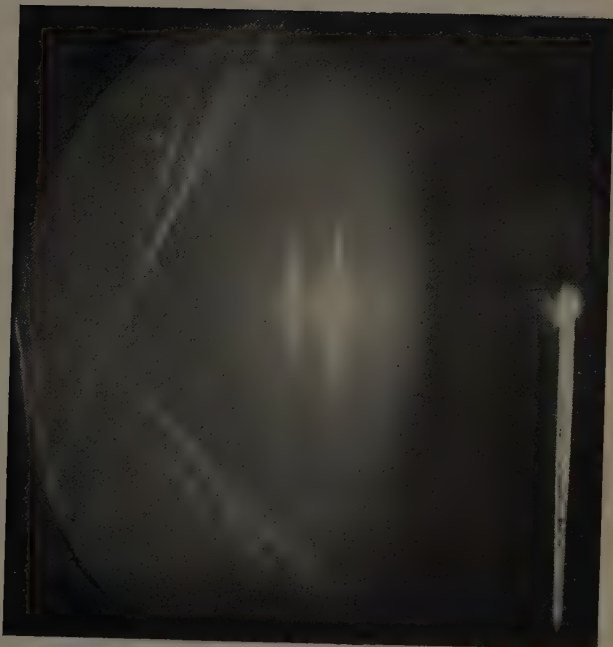
FIG. 7.



(*a*). (*b*).

Oscillograms of vowels *i* and *a* recorded at the far end of the line *.

* These oscillograms were recorded by A. Gemelli, President of the Catholic University, in Milan.



Electron diffraction pattern from (111) face of diamond.

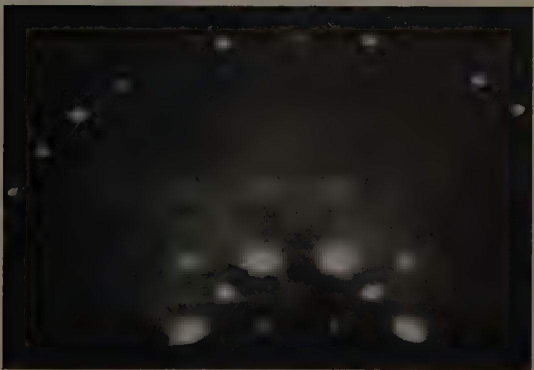
FIG. 1.

Rock salt. P = 5500 volts.

FIG. 2.

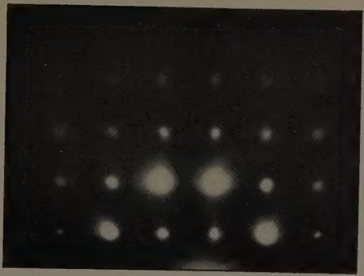
ZnS. (110) face. $\times 1{\cdot}8$.

FIG. 3.



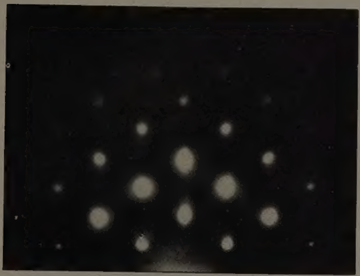
ZnS. Etched. $\times 1{\cdot}8$.

FIG. 4.



PbS. Etched. (001) azimuth. × 1·9.

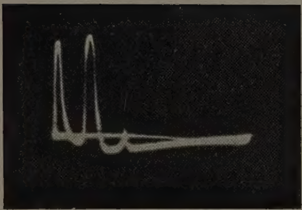
FIG. 5.



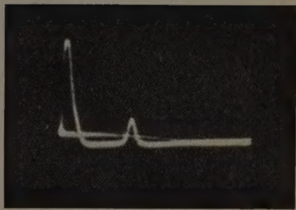
PbS. Etched. (011) azimuth. × 2·2.

FIG. 6.

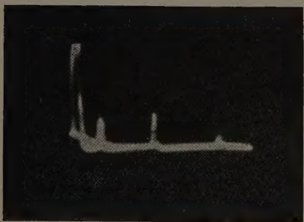
PbS. Etched. (013) azimuth. × 1·8.



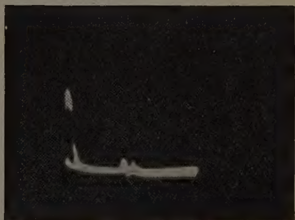
(a).



(b).



(c).



(d).

Illustrating typical echo patterns. The signal first from the left is due to the direct or ground ray ; others following are the echoes. In (a) the second and third signals are echoes once and twice reflected from the E-layer. Note the abnormal strength of the first echo. In (b) the single echo is from the F-layer. In (c) the second signal is echo from the E-layer and the third and the fourth ones are once and twice reflected echoes from the F-layer. In (d) magnetic splitting of echo from F-layer is shown.